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Contents

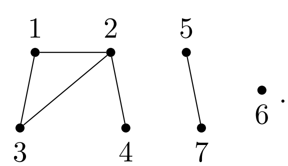
1. What is the partition algebra?

These are notes from a talk I gave in Melbourne for a group of undergraduates, Nov. 20, 2014. In particular, there are examples of how to use TikZ for graphs.

1.1. Graphs and equivalence relations.

A *graph* is a set of (labeled) vertices with adjacency relations indicated by edges. For example, one graph on 7 vertices is

(1)



An *equivalence relation* is a binary relation $\sim$ on a set X that is

reflexive,	$(x \sim x)$
symmetric, and	$(x \sim y$ implies $y \sim x)$
transitive.	$(x \sim y$ and $y \sim z$ implies $x \sim z)$

An *equivalence class* is a maximal set of pairwise equivalent elements. Given an equivalence relation on a set X, the equivalence classes *partition* the set X (meaning that every element of X is in exactly one class).

Example 1.1. *Let V be the set of vertices of a graph G. Then for $u, v \in V$,*

$$u \sim v \quad \text{if and only if there is a walk along edges from } u \text{ to } v$$

is an equivalence relation on V. The equivalence classes are the sets of vertices in the same connected components.

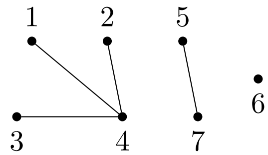
For example, in the graph (1), $V = \{1, 2, 3, 4, 5, 6, 7\}$ *and the equivalence classes are* $\{1, 2, 3, 4\}$, $\{5, 7\}$, *and* $\{6\}$.

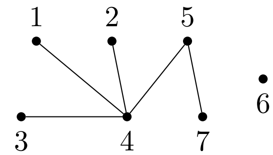
Example 1.2. *Let $\mathcal{G}$ be the set of graphs with vertices V. Then for $G, H \in \mathcal{G}$,*

$$G \sim H \quad \textit{if and only if } G \textit{ and } H \textit{ have the same connected components}$$

is an equivalence relation on $\mathcal{G}$. The equivalence classes are indexed by set partitions of V.

For example, in the graph (1) is equivalent to



, *but not to*


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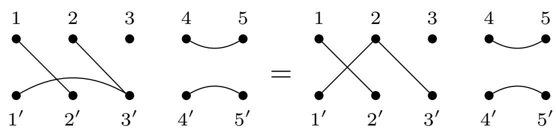
1.2. **Diagrams and their compositions.** Define a (k-)diagram as an equivalence class of graphs on vertices $V_k = \{1, 2, \ldots, k, -1, -2, \ldots, -k\}$. For example, if $k = 5$,



is the 5-diagram indexed by the partition $\{1, 2'\}, \{2, 1', 3'\}, \{3\}, \{4, 5\}, \{4', 5'\}$.

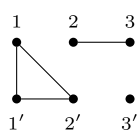
Let $D_k = \{k\text{-diagrams}\}$.

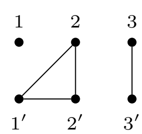
Define a multiplication

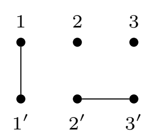
$$\circ : D_k \times D_k \to D_k$$
$$(d_1, d_2) \mapsto d_1 \circ d_2$$

by "stack d_1 on top of d_2 and resolve connections."

Example 1.3. *For example, let $k = 3$, and let*

$d_1 =$ 

, $\quad d_2 =$ 

, *and* $\quad d_3 =$ 

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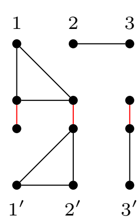
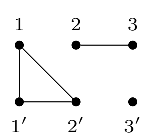
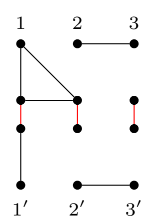
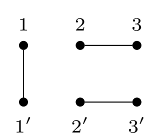
Then

$d_1 \circ d_2 =$ 

$=$ 

and $\quad d_1 \circ d_3 =$ 

$=$ 

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Things we might hope for in a multiplication:

(1) Well-defined?
(Is the multiplication independent of the choice of graph representing the diagram? Yes: Check using equivalence relation features.)

(2) Associative?
(Is $d_1 \circ (d_2 \circ d_3) = (d_1 \circ d_2) \circ d_3$? Yes: Draw some pictures, and use transitivity.)

(3) Commutative?
(No: Draw some pictures and decide)

(4) Identity?
(Yes: what is it?)

(5) Inverses?
(No: draw some pictures.)

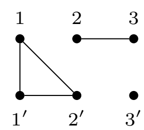
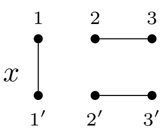
1.3. **The partition algebra.** Let $n(d_1, d_2)$ be the number of connected components lost after resolving d_1 on top of d_2 down to $d_1 \circ d_2$. For example, with d_1, d_2, d_3 as in Example 1.3,

$$n(d_1, d_2) = 0 \qquad \text{and} \qquad n(d_1, d_3) = 1.$$

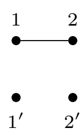
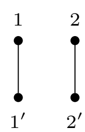
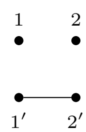
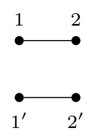
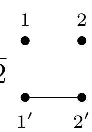
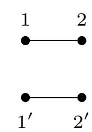
Let $\mathbb{C}D_k$ be the vector space with basis D_k. For example, there are two 1-diagrams, so $\mathbb{C}D_1 \cong \mathbb{C}^2$. Fix $x \in \mathbb{C}$. Define another multiplication, this time using n, by

$$\begin{aligned} \cdot : D_k \times D_k &\to \mathbb{C}D_k \\ (d_1, d_2) &\mapsto x^{n(d_1,d_2)} d_1 \circ d_2, \end{aligned} \tag{2}$$

and extend linearly to $\mathbb{C}D_k$ (i.e. use distributivity and linear scaling). For example, with d_1, d_2, d_3 as in Example 1.3,

$d_1 \cdot d_2 = x^0(d_1 \circ d_2) =$  and $d_1 \cdot d_3 = x^1(d_1 \circ d_3) = x$  .

Exercise 1.4. Show

$\Big(3$  $+ \sqrt{2}$  $\Big) \cdot \Big(\pi$  $- 5$  $\Big) = \pi\sqrt{-2}$  $+\big(3\pi x^2 - 15x - 5\sqrt{-2}\big)$  .

An *algebra* is a vector space equipped with a multiplication. The *partition algebra* is

$$P_k(x) = \mathbb{C}D_k \text{ with the multiplication in (2).}$$

2. Combinatorial representation theory: the symmetric group

This section is taken from some notes I made for a graduate course on combinatorial representation theory. You can see examples of how to do partitions, and how to make new commands.

Combinatorial representation theory is the study of representations of algebraic objects, using combinatorics to keep track of the relevant information. To see what I mean, let's take a look at the symmetric group.

2.1. **The symmetric group.** Let F be your favorite field of characteristic 0. Recall that an *algebra* A over F is a vector space over F with an associative multiplication

$$A \otimes A \to A$$

Here, the tensor product is over F, and just means that the multiplication is bilinear. Our favorite examples for a while will be

(1) Group algebras (today)
(2) Enveloping algebras of Lie algebras (later)

And our favorite field is $F = \mathbb{C}$.

The *symmetric group* S_k is the group of permutations of $\{1, \ldots, k\}$. The *group algebra* $\mathbb{C}S_k$ is the vector space

$$\mathbb{C}S_k = \left\{ \sum_{\sigma \in S_k} c_\sigma \sigma \mid c_\sigma \in \mathbb{C} \right\}$$

with multiplication linear and associative by definition:

$$\left(\sum_{\sigma \in S_k} c_\sigma \sigma \right) \left(\sum_{\pi \in S_k} d_\pi \pi \right) = \sum_{\sigma, \pi \in G} (c_\sigma d_\pi)(\sigma \pi).$$

Example 2.1. *When* $k = 3$,

$$S_3 = \{1, (12), (23), (123), (132), (13)\} = \langle s_1 = (12), s_2 = (23) \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle.$$

So

$$\mathbb{C}S_3 = \{c_1 + c_2(12) + c_3(23) + c_4(123) + c_5(132) + c_6(13) \mid c_i \in \mathbb{C}\}$$

and, for example,

$$\begin{aligned}(2 + (12))(5(123) - (23)) &= 10(123) - 2(23) + 5(12)(123) - (12)(23) \\ &= 10(123) - 2(23) + 5(23) - (123) = \boxed{3(23) + 9(123)}.\end{aligned}$$

2.2. **Some representations.** A *homomorphism* is a structure-preserving map. A *representation* of an F-algebra A is a vector space V over F, together with a homomorphism

$$\rho : A \to \mathrm{End}(V) = \{ \ F\text{-linear maps } V \to V \ \}.$$

The map (equipped with the vector space) is the representation; the vector space (equipped with the map) is called an A-*module.*

Example 2.2. *Favorite representation of* S_n *is the permutation representation: Let* $V = \mathbb{C}^k = \mathbb{C}\{v_1, \ldots, v_k\}$. *Define*

$$\rho : S_k \to \mathrm{GL}_k(\mathbb{C}) \qquad \text{by} \qquad \rho(\sigma) v_i = v_{\sigma(i)}$$

$k = 2$*:*

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho(\mathbb{C}S_2) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{C} \right\} \subset \mathrm{End}(\mathbb{C}^2)$$

$k = 3$:

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\rho(\mathbb{C}S_3) = \left\{ \begin{pmatrix} a+c & b+e & d+f \\ b+d & a+f & c+e \\ e+f & c+d & a+b \end{pmatrix} \;\middle|\; a, b, c, d, e, f \in \mathbb{C} \right\} \subset \mathrm{End}(\mathbb{C}^3)$$

A representation/module V is *simple* or *irreducible* if V has no invariant subspaces.

Example 2.3. *The permutation representation is not simple since $v_1 + \cdots + v_k = (1, \ldots, 1)$ is invariant, and so $T = \mathbb{C}\{(1, \ldots, 1)\}$ is a submodule (called the* trivial representation*). However, the trivial representation is one-dimensional, and so is clearly simple. Also, the orthogonal compliment of T, given by*

$$S = \mathbb{C}\{v_2 - v_1, v_3 - v_1, \ldots, v_k - v_1\}$$

is also simple (called the standard representation*). So V* decomposes *as*

$$V = T \oplus S \tag{3}$$

by the change of basis

$$\{v_1, \ldots, v_k\} \to \{v, w_2, \ldots, w_k\} \qquad \textit{where } v = v_1 + \cdots + v_k \textit{ and } w_i = v_i - v_1.$$

New representation looks like

$$\rho(\sigma)v = v, \qquad \rho(\sigma)w_i = w_{\sigma(i)} - w_{\sigma(1)} \qquad \textit{where } w_1 = 0.$$

For example, when $k = 3$,

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

Notice, the vector space $\mathrm{End}(\mathbb{C}^2)$ *is four-dimensional, and the four matrices*

$$\rho_S(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_S((12)) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$\rho_S((23)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \rho_S((132)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

are linearly independent, so $\rho_S(\mathbb{C}S_3) = \mathrm{End}(\mathbb{C}^2)$*, and so (at least for* $k = 3$*)* S *is also simple! So the decomposition in* (3) *is complete.*

An algebra is *semisimple* if all of its modules decompose into the sum of simple modules.

Example 2.4. *The group algebra of a group* G *over a field* F *is semisimple iff* $\mathrm{char}(F)$ *does not divide* $|G|$*. So group algebras over* $\mathbb{C}$ *are all semisimple.*

We like semisimple algebras because they are isomorphic to a direct sum over their simple modules of the ring of endomorphisms of those module (*Artin-Wedderburn theorem*).

$$A \cong \bigoplus_{V \in \hat{A}} \mathrm{End}(V)$$

where $\hat{A}$ is the set of representative of A-modules. So studying a semisimple algebra is "the same" as studying its simple modules.

2.3. **How combinatorics fits in.**

Theorem 2.5. *For a finite group* G*, the irreducible representations of* G *are in bijection with its conjugacy classes.*

Proof.

(A) Show

(1) the class sums of G, given by

$$\left\{ \sum_{h \in \mathcal{K}} h \mid \mathcal{K} \text{ is a conjugacy class of } G \right\}$$

form a basis for $Z(FG)$;

Example: $G = S_3$. The class sums are

$$1, \quad (12) + (23) + (13), \quad \text{and} \quad (123) + (132)$$

(2) and $\dim(Z(FG)) = |\hat{G}|$ where $\hat{G}$ is an indexing set of the irreducible representations of G.

(B) Use character theory. A *character* χ of a group G corresponding to a representation ρ is a linear map

$$\chi_\rho : G \to \mathbb{C} \qquad \text{defined by} \qquad \chi_\rho : g \to \mathrm{tr}(\rho(g)).$$

Nice facts about characters:

(1) They're *class functions* since

$$\chi_\rho(hgh^{-1}) = \mathrm{tr}(\rho(hgh^{-1})) = \mathrm{tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathrm{tr}(\rho(g)) = \chi_\rho(g).$$

Example: The character associated to the trivial representation of any group G is $\chi_1 = 1$.

Example: Let χ be the character associate to the standard representation of S_3. Then

$$\chi(1) = 2, \qquad \chi((12)) = \chi((23)) = \chi((13)) = 0, \qquad \chi((123)) = \chi(132) = -1.$$

(2) They satisfy nice relations like

$$\chi_{\rho\oplus\psi} = \chi_\rho + \chi_\psi$$
$$\chi_{\rho\otimes\psi} = \chi_\rho \chi_\psi$$

(3) The characters associated to the irreducible representations form an orthonormal basis for the class functions on G. (This gives the bijection)

Studying the representation theory of a group is "the same" as studying the character theory of that group.

This is not a particularly satisfying bijection, either way. It doesn't say "given representation X, here's conjugacy class Y, and vice versa." □

Conjugacy classes of the symmetric group are given by cycle type. For example the conjugacy classes of S_4 are

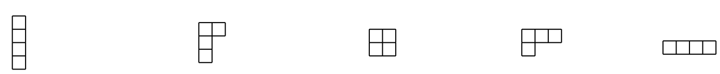
$$\begin{aligned}
&\{1\} = \{(a)(b)(c)(d)\}\\
&\{(12),(13),(14),(23),(24),(34)\} = \{(ab)(c)(d)\}\\
&\{(12)(34),(13)(24),(14)(23)\} = \{(ab)(cd)\}\\
&\{(123),(124),(132),(134),(142),(143),(234),(243)\} = \{(abc)(d)\}\\
&\{(1234),(1243),(1324),(1342),(1423),(1432)\} = \{(abcd)\}.
\end{aligned}$$

Cycle types of permutations of k are in bijection with *partitions* $\lambda \vdash k$:

$$\lambda = (\lambda_1, \lambda_2, \dots) \quad \text{with } \lambda_1 \geq \lambda_2 \geq \dots, \quad \lambda_i \in \mathbb{Z}_{\geq 0}, \lambda_1 + \lambda_2 + \cdots = k.$$

The cycle types and their corresponding partitions of 4 are

$$\begin{array}{ccccc}
(a)(b)(c)(d) & (ab)(c)(d) & (ab)(cd) & (abc)(d) & (abcd)\\
(1,1,1,1) & (2,1,1) & (2,2) & (3,1) & (4)
\end{array}$$



where the picture is an up-left justified arrangement of boxes with λ_i boxes in the ith row.

The combinatorics goes way deep! *Young's Lattice* is an infinite leveled labeled graph with vertices and edges as follows.

Vertices: Label vertices in label vertices on level k with partitions of k.
Edges: Draw and edge from a partition of k to a partition of $k+1$ if they differ by a box.

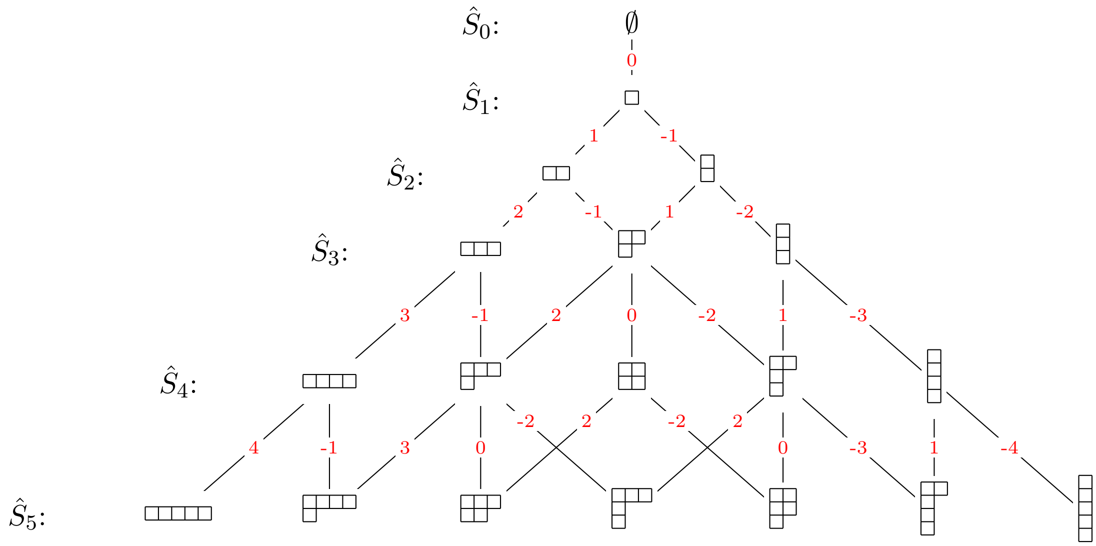
See Figure 1.

Some combinatorial facts: (without proof)

(1) The representations of S_k are indexed by the partitions on level k.
(2) The basis for the module corresponding to a partition λ is indexed by downward-moving paths from $\emptyset$ to λ.
(3) The representation is encoded combinatorially as well. Define the *content* of a box b in row i and column j of a partition as

$$c(b) = j - i, \qquad \text{the diagonal number of } b.$$

FIGURE 1. Young's lattice, levels 0–5.



Label each edge in the diagram by the content of the box added. The matrix entries for the transposition $(i\ i+1)$ are functions of the values on the edges between levels $i-1$, i, and $i+1$.

(4) If S^λ is the module indexed by λ, then

$$\mathrm{Ind}_{S_k}^{S_{k+1}}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k+1 \\ \lambda - \mu}} S^\mu \qquad \text{and } \mathrm{Res}_{S_{k-1}}^{S_k}(S^\lambda) = \bigoplus_{\substack{\mu \vdash k-1 \\ \mu - \lambda}} S^\mu$$

(where $\mathrm{Res}_{S_{k-1}}^{S_k}(S^\lambda)$ means forget the action of elements not in S_{k-1}, and $\mathrm{Ind}_{S_k}^{S_{k+1}}(S^\lambda) = \mathbb{C}S_{k_1} \otimes_{\mathbb{C}S_k} S^\lambda$).