1.1 Modeling Via Differential Equations

* Mathematical models give a mathematical representation of some real-life situation
  - Models usually change with time and may depend on other variables.
* 3 Basic Steps for Creating a Model
  1) State assumptions that explain relationships between things being studied.
  2) Describe all variables and parameters used.
     - independent variable - independent of other quantities, usually time (t)
     - dependent variable - function of the independent variable
     - parameter - quantity that does not change with the independent variable
  3) Use assumptions from (1) to derive equations relating quantities from (2)
     - Often expressed as differential equations (involves a derivative, "rate of change")
     - Make the algebra as simple as possible
* First-order equation - contains only first derivatives
* Ordinary differential equation - does not contain partial derivatives
* Equilibrium solution - a solution of a differential equation that is constant ($\frac{dP}{dt} = 0$)
* Initial condition - value of a function when $t = 0$
* Logistic population model - population growth depends on a growth rate (k) and carrying capacity (N)
  - Contrasts with an exponential growth model (population does not have a carrying capacity)
* First-order system - a system of equations - each equation contains only first derivatives, but more than one dependent variable

* Analyzing Models
1) Analytic - searches for explicit formulas that describe the behavior of the solutions
↳ Few equations have explicit solutions
2) Qualitative - uses geometry to give an overview of the model's long-term behavior
↳ Describes trends → explosions in populations, increases, decreases, etc.
3) Numerical - computer approximates a solution

1.2 Analytic Technique: Separation of Variables

* Solution of a differential equation - function of the independent variable that satisfies the differential equations for all values of the independent variable when substituted for the dependent variable
↳ $y(t)$ is a solution of $\frac{dy}{dt} = f(t,y)$ if
$$\frac{dy}{dt} = y'(t) = f(t, y(t))$$
* Initial-value problem - A differential equation and an initial value are given
↳ Allows you to solve for the constant of integration (c)
↳ General solution - c is not yet determined
* Can be used to find solution for any initial value
* Separable Equations - equations that can be written as the product of 2 functions
↳ Can be written as $\frac{dy}{dt} = g(t)\,h(y)$
* autonomous - right side of the equation depends only on dependent variable
↳ ex. $\frac{dy}{dt} = h(y)$
* Solving Separable Equations
1) Separate variables - y's on one side, t's on other side
2) "Informal" algebra - use a type of u-substitution to "multiply" both sides by dt

3) Integrate

ex. $\frac{dy}{dt} = g(t)h(y)$ → Divide by $h(y)$

$\frac{1}{h(y)}\frac{dy}{dt} = g(t)$ → "Multiply" by dt

$\frac{1}{h(y)}dy = g(t)dt$ → Integrate

$$\int \frac{1}{h(y)}dy = \int g(t)dt$$

* Missing Solutions - CHECK FOR EQUILIBRIUM SOLUTIONS
  ↳ Watch division - cannot divide by zero, but Ø may be an equilibrium solution (check original equation)

EXAMPLES from Review Section

11) True/False: The function $y(t) = -e^{-t}$ is a solution to the differential equation $\frac{dy}{dt} = |y|$.

*For a function to be a solution of a differential equation it must satisfy the equation for all values of t (independent variable) when substituted for y (dependent variable)

↳ $\frac{dy}{dt} = y'(t) = f(t, y(t))$

$\frac{dy}{dt} = |y|$ → substitute $y(t)$ for y ⟹ $\boxed{\frac{dy}{dt} = |y(t)| = e^{-t}}$

$y(t) = -e^{-t}$ → Take derivative ⟹ $\boxed{y'(t) = -e^{-t}(-1) = e^{-t}}$

$\frac{dy}{dt} = y'(t) = e^{-t}$

True

continued...

25) $\frac{dy}{dt} = 2ty^2 + 3y^2$

(a) Is the equation autonomous?

* No, the right side of the equation involves both the independent and dependent variable.

Is the equation separable?

* Yes, the right side of the equation can be written as a product of two functions.

$$\frac{dy}{dt} = y^2(2t+3)$$

(b) Find the general solution

*Beginning with factored form

$$\frac{dy}{dt} = y^2(2t+3)$$ → Divide both sides by y^2 to separate variables

$$\frac{1}{y^2}\frac{dy}{dt} = 2t+3$$ → Use "informal" algebra to multiply both sides by dt

$$\frac{1}{y^2}dy = (2t+3)dt$$ → Integrate both sides

$$\int \frac{1}{y^2}dy = \int (2t+3)dt \Rightarrow \frac{-1}{y} = t^2 + 3t + c$$ → Take the reciprocal of each side and multiply by -1 to get explicit form

$$y = \frac{-1}{t^2+3t+c}$$

$y=0$ Equilibrium Solution