Key

Classification of 2 dimensional first order homogeneous linear systems

To solve

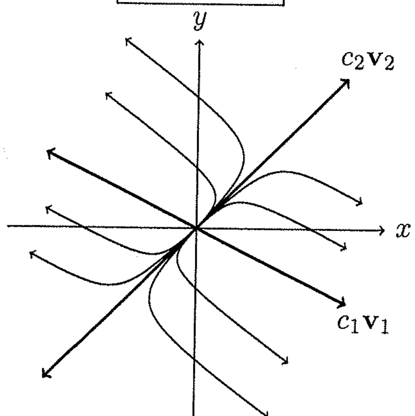
$$\frac{d\mathbf{y}}{dt} = A\mathbf{y},$$

first find eigenvalues λ_1, λ_2 (sometimes $\lambda_1 = \lambda_2$), and then find the associated $\mathbf{v}_1$ and $\mathbf{v}_2$ (sometimes if $\lambda_1 = \lambda_2$, then might only be one dimension worth of them). For each of the cases below, sketch an example and classify the equilibrium solution. I've done an example for you.

Case 1: There are two distinct eigenvalues, $\lambda_1 \neq \lambda_2$.

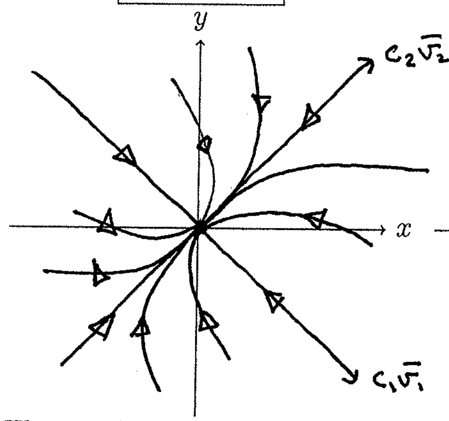
Real eigenvalues

$\lambda_1 > \lambda_2 > 0$



The equilibrium solution (0,0) is a source!

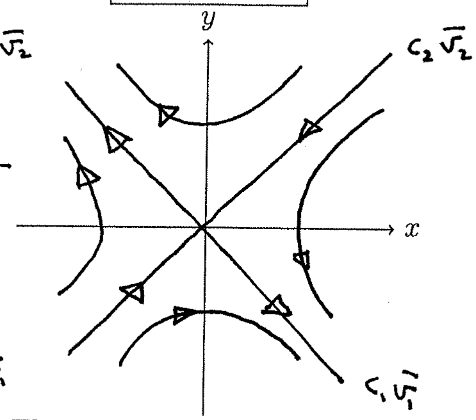
$\lambda_1 < \lambda_2 < 0$



The equilibrium solution (0,0) is a sink

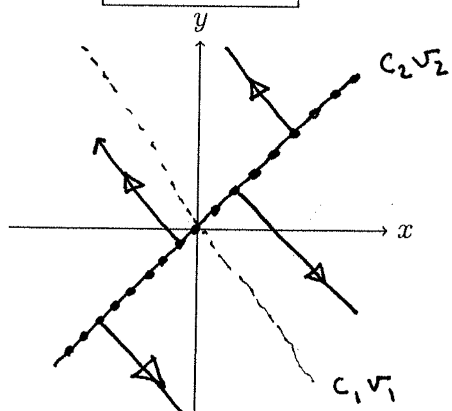
Source ↓ Sink ↓

$\lambda_1 > 0, \lambda_2 < 0$



The equilibrium solution (0,0) is a Saddle

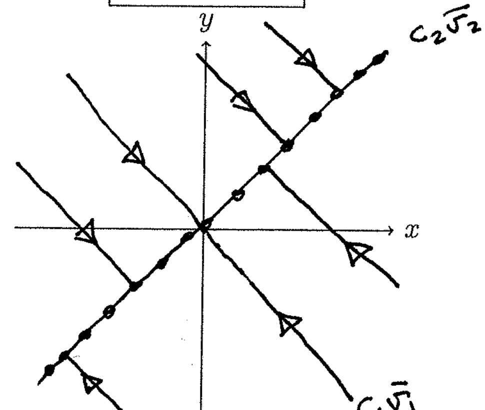
$\lambda_1 > 0, \lambda_2 = 0$



The equilibrium solutions are all pts on $c_2\bar{v}_2$
(equations)

and they are all Sources
(classification)

$\lambda_1 < 0, \lambda_2 = 0$



The equilibrium solutions are all pts on $c_2\bar{v}_2$
(equations)

and they are all sinks
(classification)

Complex eigenvalues $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$

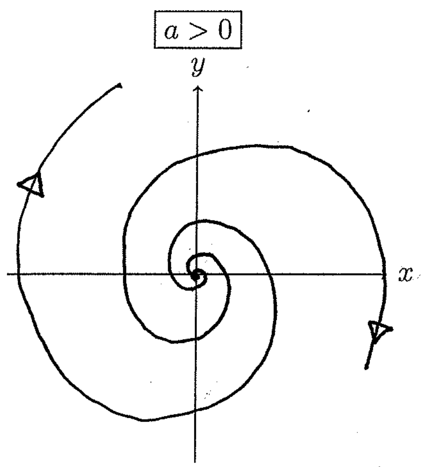
Solve just for $\mathbf{v}_1$. Then use $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to get your *real* solutions.

Every time, if $\mathbf{v}_1 = \begin{pmatrix} \alpha + i\beta \\ \gamma + i\delta \end{pmatrix}$, your calculation should look like:

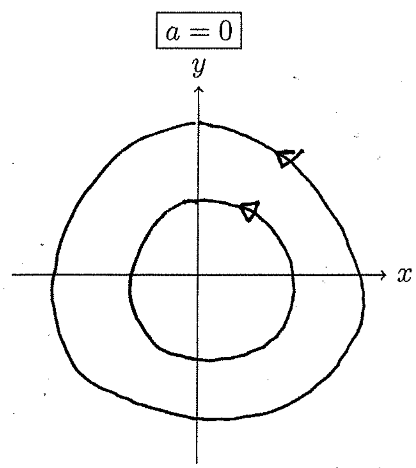
$$\begin{aligned} e^{\lambda t}\mathbf{v}_1 = e^{at+ibt}\mathbf{v}_1 &= e^{at}(\cos(bt) + i\sin(bt))\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= e^{at}\begin{pmatrix} \alpha\cos(bt) + i\alpha\sin(bt) + i\beta\cos(bt) + i^2\beta\sin(bt) \\ \gamma\cos(bt) + i\gamma\sin(bt) + i\delta\cos(bt) + i^2\delta\sin(bt) \end{pmatrix} \\ &= e^{at}\left(\begin{pmatrix} \alpha\cos(bt) - \beta\sin(bt) \\ \gamma\cos(bt) - \delta\sin(bt) \end{pmatrix} + i\begin{pmatrix} \alpha\sin(bt) + \beta\cos(bt) \\ \gamma\sin(bt) + \delta\cos(bt) \end{pmatrix}\right) \end{aligned}$$

So the general solution is $y = e^{at}(c_1\mathbf{u}_1 + c_2\mathbf{u}_2)$,

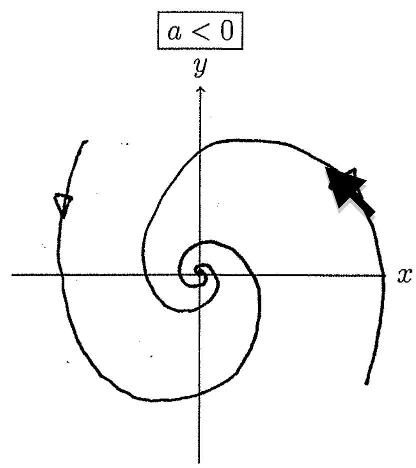
$$\text{where } \mathbf{u}_1 = \underbrace{\cos(bt)\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} - \sin(bt)\begin{pmatrix} \beta \\ \delta \end{pmatrix}}_{\text{periodic parametric curve}} \quad \text{and} \quad \mathbf{u}_2 = \underbrace{\sin(bt)\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} + \cos(bt)\begin{pmatrix} \beta \\ \delta \end{pmatrix}}_{\text{periodic parametric curve}}$$



The equilibrium solution (0,0) is a Spiral source



The equilibrium solution (0,0) is a center



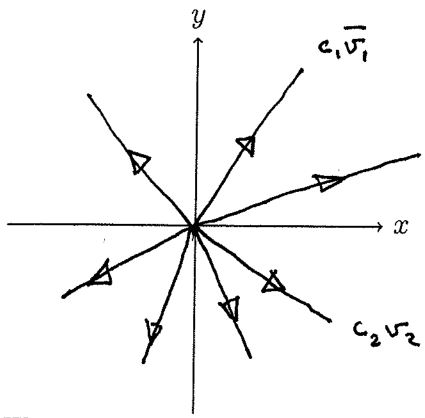
The equilibrium solution (0,0) is a Spiral sink

(can spiral either cc.wise or c.wise)

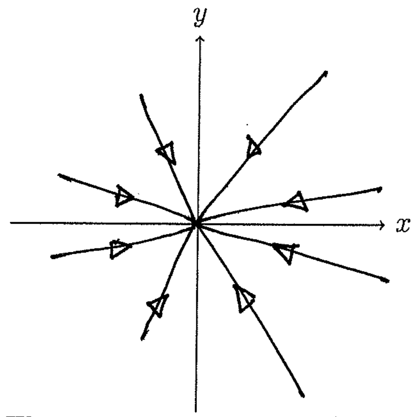
Case 2: There is one repeated eigenvalue $\lambda = \lambda_1 = \lambda_2$.
If we still get two eigenvectors, then the general solution still looks like $\mathbf{y} = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}\mathbf{v}_2$, but since the $e^{\lambda t}$ factors out of both terms, we get, simply

$$\mathbf{y} = e^{\lambda t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

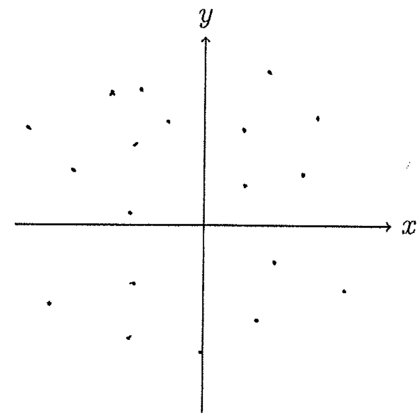
$\lambda > 0$ has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$	$\lambda < 0$ has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$	$\lambda = 0$ has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$



The equilibrium solution (0,0) is a Source



The equilibrium solution (0,0) is a Sink

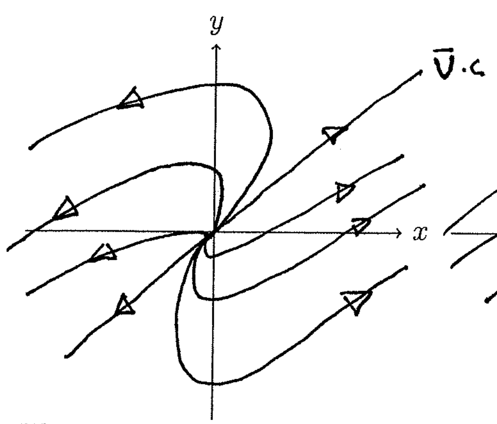


The equilibrium solutions are all pts on x-y plane

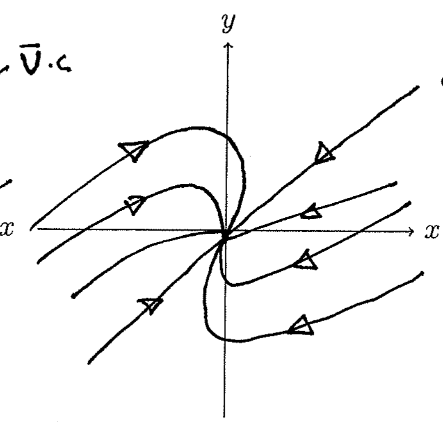
If there's only one eigenvector $\mathbf{v}$, though, we change out strategy, and get

$$\mathbf{y} = e^{\lambda t}(\mathbf{v}_0 + t\mathbf{v}_1), \qquad \text{where } \mathbf{v}_0 \text{ is free and } \mathbf{v}_1 = (A - \lambda I)\mathbf{v}_0 = c\mathbf{v}.$$

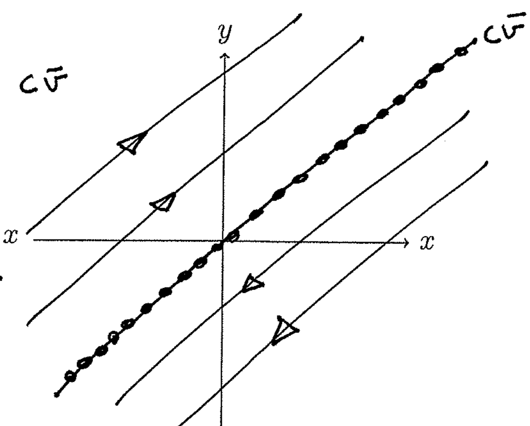
$\lambda > 0$ has one linearly independent eigenvector $\mathbf{v}$	$\lambda < 0$ has one linearly independent eigenvector $\mathbf{v}$	$\lambda = 0$ has one linearly independent eigenvector $\mathbf{v}$



The equilibrium solution (0,0) is a Source



The equilibrium solution (0,0) is a sink



The equilibrium solutions are all pts on $c\bar{v}$

(Can spiral ccwise or cwise)

(top goes op direction from bottom)