Solutions

Math A68 – Quiz 2 – Thursday 10/1/15

Instructions: justify your answers where appropriate. Leave numerical answers unsimplified, e.g. in the form 9^3, $\binom{9}{3}$, $\binom{9}{3,3,3}$, or $\left(\!\binom{9}{3}\!\right)$.

1. Give an example of a function $F(x)$ for which $e^{F(x)}$ does not formally converge.

Any $F(x)$ for which $F(0) \neq 0$, e.g. $f(x) = x+1$.

2. Use the generalized binomial theorem to give the series expansion of $\sqrt{1+5x}$.

$$\sqrt{1+5x} = (1+5x)^{1/2} = \sum_{n \in \mathbb{N}} \binom{1/2}{n} (5x)^n = \sum_{n \in \mathbb{N}} \binom{1/2}{n} 5^n x^n.$$

3. How many compositions are there of 10?

$$2^{n-1} = \boxed{2^9}$$

4. How many ways can I buy a set of 10 shirts if the store has red, blue, and black shirts to choose from? n stars, $k-1$ bars, $n=10$, $k=3$:

$$\binom{n+k-1}{k-1} = \binom{10+3-1}{3-1} = \binom{10+3-1}{10}$$

5. How many lattice paths are there from $(0,0)$ to $(1,3,6)$ with steps from $S = \{(1,0,0),(0,1,0),(0,0,1)\}$?

$$\binom{a_1+a_2+a_3}{a_1, a_2, a_3} = \boxed{\binom{1+3+6}{1,3,6}}$$

6. Recall the identity that if $S = \{x_1, \dots, x_n\}$, then $\prod_{x_i \in S}(1 + x_i) = \sum_{T \subseteq S} \prod_{x_i \in T} x_i$. Use this identity to prove the binomial theorem.

Evaluating at $x_1 = \cdots = x_n = x$, we have

$$(1+x)^n = \prod_{x_i \in S} (1+x)$$

$$= \sum_{T \subseteq S} \prod_{x_i \in T} x = \sum_{T \subseteq S} x^{|T|},$$ so collecting like terms gives

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

since there are $\binom{n}{k} = \left|\binom{S}{k}\right|$ subsets of S of size k, for $k = 0, \dots, n$.

7. Show that the number a_n of compositions of n into parts greater than 1 is satisfies $a_n = a_{n-1} + a_{n-2}$ for $n \geqslant 2$.

Let $S_n = \{$ good comps of $n\}$, $T_n = \{(\alpha_1, \dots, \alpha_\ell) \in S_n \mid \alpha_\ell = 2\}$,

and $U_n = \{(\alpha_1, \dots, \alpha_\ell) \in S_n \mid \alpha_\ell > 2\}$, so that

$S_n = T_n \sqcup U_n$.

Note that for $n > 2$,

$$\begin{aligned} T_n &\longrightarrow S_{n-2} \\ (\alpha_1, \dots, \alpha_\ell) &\longmapsto (\alpha_1, \dots, \alpha_{\ell-2}) \end{aligned} \quad \text{and} \quad \begin{aligned} U_n &\longrightarrow S_{n-1} \\ (\alpha_1, \dots, \alpha_\ell) &\longmapsto (\alpha_1, \dots, \alpha_\ell - 1) \end{aligned}$$

are both bijections, so

$$a_n = |S_n| = |T_n| + |U_n| = |S_{n-2}| + |S_{n-1}|$$

$$= a_{n-2} + a_{n-1}.$$

$\boxed{4/}$