Math 680 – Quiz 1 – Tuesday 9/8/15

Instructions: justify your answers where appropriate. Leave numerical answers unsimplified.

1. How many subsets of the set [10] contain at least one odd integer?

\# subsets of [10] : 2^{10}

\# subsets w/ no odd integers

$= \#$ subsets of $\{2,4,6,8,10\} = 2^5$

Total : $\boxed{2^{10}-2^5.}$

2. Recall that a *cycle* of a permutation is a sequence $(c_1, c_2, \ldots, c_\ell)$ such that

$$w : c_1 \mapsto c_2, \quad w : c_2 \mapsto c_3, \quad \ldots \quad w : c_{\ell-1} \mapsto c_\ell, \quad w : c_\ell \mapsto c_1.$$

How many permutations $w : [6] \to [6]$ have exactly one cycle?

If there is one cycle, it has 1 in it. Without loss of generality let $c_1 = 1$. Then a cycle

$1 \to c_2 \to \cdots \to c_6 \to 1$

is equivalent to (is in bijection with) a permutation $c_2, c_3, \ldots, c_6$ of $[6]-\{1\}$, of which there are $\boxed{5!}$.

3. Give both the generating and exponential generating functions for $f(n) = 5^n$. For each, give your answer in series form. If possible, also give your answer in closed form.

Gen fn: $\sum_{n \in \mathbb{N}} 5^n x^n = \frac{1}{1-5x}$.

Exp gen fn: $\sum_{n \in \mathbb{N}} \frac{5^n x^n}{n!} = e^{5x}$.

4. Fill in the the following combinatorial proof of the identity

$$\binom{n}{k} = \frac{(n)_k}{k!}, \qquad \text{where } (n)_k = n(n-1)\cdots(n-(k-1)).$$

Proof. Let $N(n,k)$ be the number of ways to choose a size-k subset T of $[n]$, together with a linear ordering of T.

(a. Showing $N(n,k) = \binom{n}{k}k!$) On the one hand...

we can first pick T, of which there are $\binom{n}{k}$ ways, and then pick the ordering, of which there are $k!$ ways. Then combine using the product rule.

Therefore, $N(n,k) = \binom{n}{k}k!$.

(b. Showing $N(n,k) = (n)_k$) On the other hand...

we can pick the elements one at a time in order, of which there are

$n(n-1) \cdots (n-(k-1)) = (n)_k$ ways.

Therefore, $N(n,k) = (n)_k$.

(c. Conclusion) Therefore...

$$\binom{n}{k}k! = N(n,k) = (n)_k.$$

Since $k! \neq 0$ for $0 \le k \le n$, we can divide both sides by $k!$.

Thus

$$\binom{n}{k} = \frac{(n)_k}{k!}.$$

□