Exam #1

Math A68 – Daugherty

October 20, 2015

Name (Print): ____
First   Last

Instructions: You are not allowed to use calculators, books, or notes of any kind in aid of completing this exam. Justify/explain all of your answers; answers without justification will not receive full credit. If you are asked to explain, use words. If you need more space, there is an extra page at the end; if you need even more, please ask for additional paper. If you get stuck, you may ask for hints, put you may lose points; if you need an answer to an earlier part to finish a problem, but are unable to complete that earlier part, I can provide you with the answer for the price of those points. Feel free to leave answers unsimplified when appropriate, e.g. $2 * 3!$ or $\binom{3}{1,1,1}$.

Please sign indicating you have read these instructions.

Signature:___________________________________

Problem #	Out of	Score
1	12	
2	13	
3	27	
4	13	
5	10	
6	10	
7	15	
total	100	

1. (a) How many subsets of [6] contain at least one even integer?

(b) How many ways can you choose 6 bagels to buy from a bakery that stocks plain, raisin, and onion?

(c) How many ways can you deal out a 52-card deck so that each person gets exactly 13 cards?

(d) How many ways are there to walk to the point $(7, 2, 5, 1)$ from the origin $(0, 0, 0, 0)$ if you are allowed to take elementary unit steps (i.e. steps in the direction $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, and $e_4 = (0, 0, 0, 1)$)?

2. Let $w = 52134$.

(a) Write w in standard cycle notation.

(b) What is the descent set $D(w)$?

(c) What is the major index $\text{maj}(w)$?

(d) What is the cycle type $\text{type}(w)$?

(*Still*, $w = 52134$)

(e) What is the inversion table $I(w)$?

(f) Check your answer to part (2e) by recursively constructing w from its inversion table. (Recall: go in decreasing order, inserting i so that it loses in a_i inversions.)

3. (a) Give the formula for the number of permutations of $[n]$ that have cycle type $(c_1, c_2, \ldots, c_n)$, given that $\sum_i ic_i = n$.

(b) List the 5 possible cycle types of permutations of $\mathcal{S}_4$. Then use elementary counting methods (product, sum, division rules, binomial coefficients) to count the number of permutations of $[4]$ with each cycle type; justify your numerical answers. Finally, check your answer using the formula in part (a). Type $(4, 0, 0, 0)$ is given as an example.

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type 1: $\boxed{(4, 0, 0, 0)}$

number of permutations of type 1 (elementary methods): check:

$\boxed{\dfrac{\#\{\text{ways to choose the four 1-cycles}\}}{\#\{\text{permutations of the 1-cycles}\}} = 4!/4! = 1}$ $\boxed{4!/1^4 4! = 1}$

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type 2:

number of permutations of type 2 (elementary methods): check:

type 3:

number of permutations of type 3 (elementary methods): check:

(Continued on next page)

(3(b) continued from last page)

type 4:

number of permutations of type 4 (elementary methods): check:

type 5:

number of permutations of type 5 (elementary methods): check:

(c) Use your answers in part (b) to

(i) give the cycle indicator function Z_4 (the generating function related to cycle type);

(ii) give the generating function for the signless Stirling numbers of the first kind $c(n, k)$ for $n = 4$ (the generating function related to the number of cycles).

(d) Explain why, <u>for general n</u>, the generating function for $c(n, k)$ is the same as $n!Z_n(t, t, \ldots, t)$.

4. (a) Define a *composition* of n with ℓ parts.

(b) Explain how to use stars and bars pictures to represent *compositions* and then to represent *weak compositions.*

(c) Use stars and bars to explain why there are 2^{n-1} compositions of n.

(d) Use stars and bars to explain why there are $\binom{n+\ell-1}{\ell-1}$ weak compositions of n with ℓ parts.

5. Let $w \in \mathcal{S}_n$. Prove that if $w = \hat{w}$, then, when written in standard cycle notation, the first cycle of w is either (1) or (21).

6. Prove by induction that, for $n \in \mathbb{N}$ and $S = \{x_1, x_2, \ldots, x_n\}$,

$$\prod_{x_i \in S} (1 + x_i) = \sum_{T \subseteq S} \prod_{x_i \in T} x_i.$$

7. Consider the identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0, \qquad n \in \mathbb{N}. \tag{$*$}$$

(a) Give a combinatorial proof of $(*)$.
[Hint: move all the negative terms to the right.]

(b) Prove $(*)$ (previous page) using the binomial theorem.

(extra space)