Combinatorial Analysis – 12/03/15

Exercise 39. (a) Compute the following.

(i) S(3,0), (ii) S(3,1), (iii) S(3,2), (iv) S(3,3),
(v) S(n,0), (vi) S(n,1), (vii) S(n, n-1), (viii) S(n,n).

(b) Give a combinatorial proof that the Sterling numbers of the second kind satisfy the recurrence relation

$$S(n,k) = kS(n-1,k) + S(n-1,k-1).$$

(c) Verify the identity

$$\sum_{k\geq 0} S(n,k)s(k,m) = \delta_{m,n}$$

for $n = 3$ and $m = 3$ and $m = 1$. (Recall $s(a,b) = (-1)^{a-b}c(a,b)$ is the Stirling number of the first kind.)

(d) Lunch box examples. Ok to give your answers in terms of $\binom{n}{m}$ or $S(n,m)$ where appropriate.

- (i) How many ways are there to distribute 6 distinct candy bars into 4 identical lunch boxes so that every lunch box gets at least one candy bar?
- (ii) How many ways are there to distribute 6 distinct pieces of fruit into 4 identical lunch boxes? (you might leave some empty)
- (iii) How many ways are there to distribute 6 identical juice boxes into 4 identical lunch boxes? (you might leave some empty)
- (iv) How many ways are there to distribute 6 identical sandwiches into 4 identical lunch boxes so that every box gets at least one sandwich?
- (v) How many ways are there to distribute 6 identical carrots into 4 distinct lunch boxes so that every box gets at least one carrot?
- (vi) How many ways are there to distribute 6 identical bottles of water into 4 distinct lunch boxes? (you might leave some empty)
- (vii) How many ways are there to distribute 6 distinct cookies into 4 distinct lunch boxes? (you might leave some empty)
- (viii) How many ways are there to distribute 6 distinct pieces of cheese into 4 distinct lunch boxes so that every box gets a piece of cheese?

Exercise 40. The following argument is very similar to the one that we used to compute $S(n,m)$, and *seems* to be an easier approach to computing the number of ways of putting n distinguishable balls into m indistinguishable boxes. However, it results in the claim that there are $m^n/m!$ such ways, which cannot possibly be true, since $m^n/m!$ is not always an integer (e.g. let $n = 2$ and $m = 3$). Find and explain the flaw in this argument.

> *To count the ways of putting n distinguishable balls into m indistinguishable boxes, first choose an ordering of the m boxes (so that they're now distinguishable). Then an assignment of n distinguishable balls into those m boxes determines a function* $\phi : [n] \to [m]$*. Since there are* m^n *such functions, and m! permutations of the m boxes, there are* $m^n/m!$ *ways of putting n distinguishable balls into m indistinguishable boxes.*

Exhibit the flaw for $n = 2$ and $m = 3$. [Hint: consider the empty boxes. If $n = 2$ and $m = 3$ doesn't make it clear to you, try other examples, but note that you need $m \geq 3$ to see what goes wrong. If you try bigger examples, focus on maps where the image has size $|f([n])| \leq m - 2$.]

Exercise 41. Read and summarize Example 2.2.4 (up to equation (2.15)), taking time to read the first page of Section 1.4.