**Exercise 38.** (a) *Derangements.* Recall that a derangement of $n$ is a permutation of $n$ with no fixed-points, and that the number of derangements of $n$ is given by

$$D(n) = n! \sum_{i=0}^{n} (-1)^i / i!.$$

(i) Verify this formula for $D(n)$ for $n = 3$.

(ii) Verify the recursive formula $D(n) = nD(n-1) + (-1)^n$ using the above formula for $D(n)$.

(iii) Give a combinatorial proof for the recursive formula $D(n) = (n-1)(D(n-1) + D(n-2))$.

(b) *Fixed-point free functions* Consider the set of functions $\varphi : [n] \to [n]$. Note that this differs from our work on derangements, since $\varphi$ is not necessarily bijective.

(i) Let $S$ be the set of conditions "$\varphi(i) = i$" (so that $|S| = n$, one condition for each element of $[n]$.) For $T \subseteq S$, describe $f_=(T)$ and $f_\geq(T)$ using set notation.

(ii) How many functions $\varphi : [n] \to [n]$ have no fixed points?

(iii) Let $E(n)$ be the number of fixed-point free functions $\varphi : [n] \to [n]$. Show that

$$\lim_{n\to\infty} E(n)/n^n = 1/e.$$

(c) How many permutations of $[n]$ have no cycle of length $k$? If $f_k(n)$ denotes this number, then compute $\lim_{n\to\infty} f_k(n)/n!$.

[Hint: for a subset $S \subseteq [n]$ of size $k$, let $A_S$ be the set of permutations in which there is a $k$-cycle whose entries are the elements of $S$. It may or may not be useful to note that $A_S$ and $A_T$ are disjoint exactly when $S$ and $T$ are not disjoint.]