Exercise 25. (a) Write the permutation matrices for each $w \in \mathcal{S}_3$.

(b) Verify that there are 14 of permutations of 4 that have no decreasing sequences of length 3 by counting the corresponding lattice paths, and listing the corresponding permutations.

(c) For each of the following pairs (u, w), decide whether or not w is u-avoiding.
  - (i) $u = 132$, $w = 7421365$;
  - (ii) $u = 132$, $w = 5671234$;
  - (iii) $u = 1234$, $w = 1765423$;
  - (iv) $u = 1234$, $w = 73164258$;
  - (v) $u = 1234$, $w = 123$.

(d) Draw the dot diagrams for each $w \in \mathcal{S}_3$.

(e) Inverses of dot diagrams.
  - (i) Verify that $D_w^t = D_{w^{-1}}$ for $w \in \mathcal{S}_3$.
  - (ii) Explain why $D_w^t = D_{w^{-1}}$.

(f) Inversion tables from dot diagrams.
  - (i) Verify that the inversion tables for the permutations $w \in \mathcal{S}_3$ can be read off of the dot diagrams D_w.
  - (ii) Explain why the inversion table $I = (a_1, \ldots, a_n)$ corresponding to a permutation w is given by

$$a_i = \#\{ \text{ circled dots in column } i \text{ of } D_w \ \}.$$

(g) Verify that the 132-avoiding permutations of [3] are Ferrers diagrams, and in bijection with the set of partitions of $0, 1, \ldots, \binom{3}{2}$ satisfying $\lambda_i \leq n - i$.

(h) Verify Proposition 1.5.1, i.e. that

$$\sum_{w \in \mathcal{S}_{132}(n)} q^{\mathrm{inv}(w)} = \sum_{\lambda} q^{|\lambda|},$$

for $n = 3$

Exercise 26. Go to

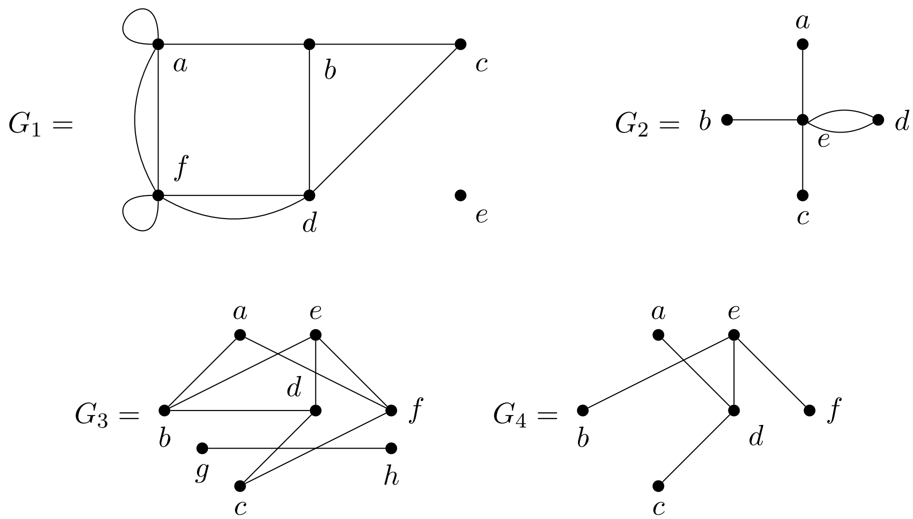
http://www-math.mit.edu/∼rstan/ec/catalan.pdf

and

http://www-math.mit.edu/∼rstan/ec/catadd.pdf

to see the famous Catalan numbers problem (6.19 in EC2). Skim the whole thing. Pick three unrelated parts and verify that the corresponding combinatorial set is of size 14 for $n = 4$. Unrelated means that if you do one part concerning lattice paths, then the other parts you pick should not concern lattice paths. Parts (h) and (i) are equivalent to the lattice paths you did in problem 25(b), so don't repeat these. You may need to look up or ask for definitions, but examples for $n = 3$ are almost always given.

Exercise 27. Let



(a) For each of the graphs G_1, G_2, G_3, and G_4, answer the following questions.
   (i) Is G_i simple? Is G_i a multigraph? Is G_i connected? Is G_i a forrest? Is G_i a tree?
   (ii) What is the neighborhood of a? of e? of $\{a, e\}$?
   (iii) What is the degree of a? of e?
   (iv) Verify the handshake theorem.
   (v) Verify that there are an even number of vertices of odd degree.

(b) List three cycles in G_3.
(c) Give an example of a closed walk in G_1 that is not a cycle.
(d) Give an example of a path of length 3 in G_3.
(e) Give an example of a maximal path in G_4.
(f) Give a walk of length 5 in G_2 that is not closed, not a path, and not a trail.
(g) Give a trail of length 8 in G_1.
(h) What's the longest length of the trails in G_2?