Exercise 24.

(a) Eulerian numbers.

(i) Verify that $A_3 = x + 4x^2 + x^3$ and $A_4 = x + 11x^2 + 11x^3 + x^4$.

(ii) In general, for $d > 0$, what are $A(d, 1)$ and $A(d, d)$?

(iii) What is $A_d(1)$?

(b) Excedances.

(i) Show that $w = w_1 w_2 \cdots w_d$ has k weak excedances if and only if $u = u_1 u_2 \cdots u_d$, defined by $u_i = d + 1 - w_{d-i+1}$, has $d - k$ excedances.

(ii) Show w has $d - 1 - j$ descents if and only if $w_d w_{d-1} \cdots w_1$ has j descents.

(iii) Show that

$$A(d, k + 1) = |\{w \in \mathcal{S}_d \mid w \text{ has } k \text{ excedances}\}|$$

and

$$A(d, k + 1) = |\{w \in \mathcal{S}_d \mid w \text{ has } k + 1 \text{ weak excedances}\}|.$$

(c) Complete the proof of Prop. 1.4.6 by proving that $\mathrm{inv}(\gamma_k) = \mathrm{maj}(\eta_k)$ implies $\mathrm{inv}(\gamma_{k+1}) = \mathrm{maj}(\eta_{k+1})$ in the case where the last letter w_k of γ_k is smaller than w_{k+1}.