Exercise 19.

(a) For $Z_n = \frac{1}{n!}\sum_{w\in\mathcal{S}_n} t^{\text{type}(w)}$, calculate Z_1, Z_2, Z_3, and Z_4 explicitly (verifying the formulas between (1.25) and (1.26) in EC1).

(b) For $E_k(n) = \frac{1}{n!}\sum_{w\in\mathcal{S}_n} c_k(w)$, verify that

$$E_k(n) = \frac{\partial}{\partial t_k} Z_n(t_1, t_2, \ldots, t_n)\Big|_{\substack{t_i=1\\ i=1,\ldots,n}}.$$

(c) Give a combinatorial proof of $E_k(n) = 1/k$ by (i) explaining why there are $\binom{n}{k}(k-1)!$ k-cycles, (ii) explaining why each k-cycle appears in $(n-k)!$ permutations, and (iii) computing $E_k(n)$ using these two values.

Exercise 20. (Alternate)

(a) Compute the signless Stirling numbers of the first kind $c(n,k)$ for $n = 1,2,3,4$ and $k = 1,\ldots,n$ (i) directly, and (ii) using the recursion. Then give the Stirling numbers of the first kind $s(n,k)$ for $n = 1,2,3,4$ and $k = 1,\ldots,n$.

(b) Verify $\sum_{k=0}^{n} c(n,k)t^k = t(t+1)(t+2)\cdots(t+n-1)$ for $n = 0,1,2,3,4$.

Exercise 21. (Proving Proposition 1.3.7)

(a) Verify that $\sum_{k=0}^{n} c(n,k)t^k = n!Z_n(t,t,\ldots,t)$ for $n = 1,2,3$, and then explain why this identity holds in general.

(b) Cary out another example for the third proof of Proposition 1.3.7, again for $n = 9$ and $k = 4$.

(c) Walk through and complete the third proof of Proposition 1.3.7.

(d) Read the fourth proof and example 1.3.9. Cary out another example for $n = 9$ and $k = 4$ for a sequence $(a_1,\ldots,a_n)$ of your choice.

Exercise 22. Using only the combinatorial definitions of the signless Stirling numbers $c(n,k)$, give formulas for $c(n,1)$, $c(n,n)$, $c(n,n-1)$, and $c(n,n-2)$.