Exercise 16. For each of the following permutations of [9], give whichever of the following is not already given.

(i) The function representation.
(ii) The word representation.
(iii) The standard cycle representation.
(iv) The digraph representation.
(v) The word given by the fundamental bijection (the ˆ word).
(vi) The diagrammatic representation.

(a) $w : [9] \to [9]$, the permutation given by

$$1 \mapsto 8, \quad 2 \mapsto 9, \quad 3 \mapsto 7, \quad 4 \mapsto 4, \quad 5 \mapsto 6, \quad 6 \mapsto 5, \quad 7 \mapsto 1, \quad 8 \mapsto 3, \quad 9 \mapsto 2.$$

(b) $v = (6325)(1)(9478)$.
(c) The permutation u determined by $\hat{u} = 123456798$.

Exercise 17. For each of the permutations in 16, give the cycle type, and the number of permutations of [9] that have the same cycle type. Additionally, give the following.

(a) For w in 16(a), verify the equation $n = \sum_i ic_i$.
(b) For v in 16(b), verify that v has the same number of cycles as $\hat{v}$ has left-to-right maxima (be sure to identify the left-to-right maxima).
(c) For u in 16(c), what is $t^{\text{type}(u)}$?

Exercise 18. Show that the number of permutations $w \in \mathcal{S}_n$ fixed by the fundamental bijection $\mathcal{S}_n \xrightarrow{\hat{}} \mathcal{S}_n$ (i.e. $|\{w \in \mathcal{S}_n \mid \hat{w} = w\}|$) is the Fibonacci number F_{n+1}.

Exercise 19.

(a) For $Z_n = \frac{1}{n!}\sum_{w \in \mathcal{S}_n} t^{\text{type}(w)}$, calculate Z_1, Z_2, Z_3, and Z_4 explicitly (verifying the formulas between (1.25) and (1.26) in EC1).
(b) For $E_k(n) = \frac{1}{n!}\sum_{w \in \mathcal{S}_n} c_k(w)$, verify that

$$E_k(n) = \frac{\partial}{\partial t_k} Z_n(t_1, t_2, \ldots, t_n)\Big|_{\substack{t_i = 1 \\ i=1,\ldots,n}}.$$

(c) Give a combinatorial proof of $E_k(n) = 1/k$ by (i) explaining why there are $\binom{n}{k}(k-1)!$ k-cycles, (ii) explaining why each k-cycle appears in $(n-k)!$ permutations, and (iii) computing $E_k(n)$ using these two values.