Combinatorial Analysis – 9/25/15

Instructions:

1. Get into groups of 2–3 people and read intro and the solution to (a)(i).
2. As a group: spend 30 minutes sketching solutions to Exercise 15.
3. On your own: pick a part to write up. Spend no more than 15 minutes writing the solution carefully. If you finish early, start on the next part, and so on.
4. Amongst you group: trade/rotate solutions. Read your group-mate's solution. Evaluate it for clarity of writing. Where could they be more clear? Are they using complete sentences? Where could they say more? Where could they be more concise? Where would "displayed equations" be helpful? Last on your list: is the solution mathematically accurate? Spend 5 minutes marking up the solution.
5. Pass back to the author to read and process revisions. Spend 5 minutes.
6. Repeat steps 3–5 until you run out of group-sketched solutions.
7. When you run out of group-sketched solutions, go back to sketching solutions as a group.

Intro: A *recurrence relation* for a sequence $(a_n)_{n\in\mathbb{N}}$ is an identity determining the nth term from the previous terms. For example, the *Fibonacci recurrence* is

$$a_n = a_{n-1} + a_{n-2}. \qquad (*)$$

Any recurrence relation needs *initial conditions* to determine the corresponding sequence. For example, with the Fibonacci recurrence,

the initial conditions $a_0 = 0, a_1 = 3$ determines the sequence $0, 3, 3, 6, 9, 15, \ldots$;

the initial conditions $a_0 = 1, a_1 = -2$ determines the sequence $1, -2, -1, -3, -4, -7, \ldots$.

The *Fibonacci numbers* are defined recursively by $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$. So $f_2 = 0 + 1 = 1$, $f_3 = 1 + 1 = 2$, $f_4 = 2 + 1 = 3$, and so on. Note that any sequence that satisfies $(*)$, and that has initial conditions consisting of two consecutive Fibonacci numbers, can be expressed in terms of f_i's (see the example below).

Exercise 15.

(a) For each of the following, give examples for small values of n. Then express the following numbers in terms of the Fibonacci numbers.

(i) **Example:** The number of subsets S of the set $[n] = \{1, 2, \ldots, n\}$ such that S contains no two consecutive integers.

Answer: Let a_n be the number of good subsets of $[n]$. Note that $a_1 = |\{\emptyset, \{1\}\}| = 2$ and $a_2 = \{\emptyset, \{1\}, \{2\}\} = 3$.

Now divide S into 2 cases: either it contains n or it doesn't. Since every good subset without n is also a good subset of $[n-1]$, and vice versa, the number of good subsets without n is a_{n-1}. Similarly, since $S \mapsto S - \{n\}$ is a bijection between good subsets of $[n]$ containing n and good subsets of $[n-2]$, the number of good subsets of $[n]$ containing n is a_{n-2}. So $a_n = a_{n-1} + a_{n-2}$, with $a_1 = 2 = f_3$, $a_2 = 3 = f_4$. This is the same recurrence that determines f_n, but shifted so that $a_n = f_{n+2}$.

So there are $\boxed{f_{n+2}}$ good subsets of $[n]$. $\square$

NOTE: For many of these, this is the strategy you want. Make a recurrence relation that looks like the Fibonacci recurrence, and shift appropriately. For at least one, you'll want to use a previous part.

(ii) The number of compositions of n into parts greater than 1.
(iii) The number of compositions of n into parts equal to 1 or 2.
(iv) The number of compositions of n into odd parts.
(v) The number of sequences $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of 0s and 1s such that $\varepsilon_1 \leq \varepsilon_2 \geq \varepsilon_3 \leq \varepsilon_4 \geq \cdots$.
(vi) The number of sequences $(T_1, T_2, \ldots, T_k)$ of subsets T_i of $[n]$ such that $T_1 \subseteq T_2 \supseteq T_3 \subseteq T_4 \supseteq \cdots$.
(vii) The sum $\sum \alpha_1 \alpha_2 \cdots \alpha_\ell$ over all 2^{n-1} compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of n.
[Hint: this sum counts the number of ways of inserting at most one vertical bar in each of the $n-1$ spaces between stars in a line of n stars, and then circling one star in each compartment. Now try replacing bars, un-circled stars, and circled stars by 1's, 2's, and 1's, respectively. Use a previous part.]

(b) Consider the identity

$$F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}.$$

(i) Check this identity for $n = 2$ and 3.
(ii) Prove this identity recursively by showing that it satisfies the Fibonacci recurrence, and that it holds for the first 2 values.
(iii) Prove this identity combinatorially. Namely, first show combinatorially that the number of k-subsets of $[n-1]$ containing no two consecutive integers is $\binom{n-k}{k}$, and then use (i).

(c) Note that EC1, Example 1.1.12 to computes the generating function for the sequence $(a_i)_{i \in \mathbb{N}}$, where $a_i = f_{i+1}$ (so $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3$, and so on). Repeat this computation for $(f_i)_{i \in \mathbb{Z}_{>0}}$, making the appropriate changes to accommodate the shift.