Combinatorial Analysis – 9/24/15

See **Exercise 13 from last time.

Exercise 14.

(a) **Deriving multinomial coefficients algebraically.** Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a composition of n. Use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to compute $\binom{n}{\alpha_1,\ldots,\alpha_\ell}$, noting that you can first choose the α_1 items from n, then α_2 from $n-\alpha_1$, then α_3 from $n-(\alpha_1+\alpha_2)$, and so on.

(b) **Multinomial theorem.** Following our proof of the binomial theorem, show that

$$(x_1 + x_2 + \cdots + x_\ell)^n = \sum_{\substack{(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{N}^\ell \\ \alpha_1+\cdots+\alpha_\ell=n}} \binom{n}{\alpha_1,\ldots,\alpha_\ell} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}.$$

[Hint: Recall that the key computation for the binomial theorem was that, for $S = \{x^{(1)}, \ldots, x^{(n)}\}$,

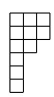
$$\prod_{x^{(i)}\in S} (1 + x^{(i)}) = \sum_{T\subseteq S} \prod_{x^{(i)}\in T} x^{(i)}, \quad \text{so that} \quad (1+x)^n = \sum_{T\subseteq S} \prod_{x^{(i)}\in T} x = \sum_{T\subseteq S} x^{|T|}.$$

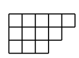
The former we had to prove by induction on n. Now fix ℓ, and let $S = \{x_i^{(j)} \mid 1 \le i \le \ell, 1 \le j \le n\}$ (so that there are n distinct variables associated to each x_i), and walk through a similar proof.]

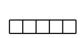
(c) **Lattice paths.** Proposition 1.2.1 in EC1 says the following. Let $v = (a_1, \ldots, a_d) \in \mathbb{N}^d$, and let e_i denote the ith unit coordinate vector in $\mathbb{Z}^d$. The number of lattice paths in $\mathbb{Z}^d$ from the origin $(0, 0, \ldots, 0)$ to v with steps in $\{e_1, \ldots, e_d\}$ is given by the multinomial coefficient $\binom{a_1+\cdots+a_d}{a_1,\ldots,a_d}$.

(i) Check this proposition for $d = 2$ with the point $v = (2, 3)$.
(ii) Check this proposition for $d = 3$ with the point $v = (1, 1, 2)$.
(iii) Prove this theorem (spell out the book's proof with more details).

(d) **Integer partitions.** An (integer) partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n is a composition of n satisfying $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell > 0$. We draw partitions as n boxes piled up and to the left into a corner, with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. For example,

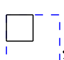
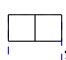
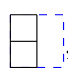
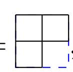
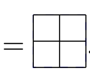
$\lambda = (3, 3, 2, 1, 1, 1) =$  is a partition of 11,

$\lambda = (5, 4, 3) =$  is a partition of 12,

$\lambda = (5) =$  is a partition of 5, and

$\lambda = \emptyset$ is a partition of 0.

The six partitions to fit in a 2×2 square are

$\emptyset =$ , $(1) =$ , $(2) =$ , $(1,1) =$ , $(2,1) =$ , and $(2,2) =$ .

Use lattice paths to count the number of integer partitions fitting into a $m \times n$ rectangle.