Exercise 9.

(a) (i) For each composition α of 3, give S_α.

(ii) Prove that the map $\theta : \alpha \to S_\alpha$ is a bijection between ℓ-compositions of n and $(\ell-1)$-subsets of $[n-1]$.
[Hint: define θ^{-1} and show that it is well-defined.]

(b) Describe a bijection between weak ℓ-compositions of n and arrangements of stars and bars, and conclude how many weak ℓ-compositions there are of n.

(c) Give a bijection between E_n, the set of compositions of n with an even number of even parts, and O_n, the set of compositions of n with an odd number of even parts.
[For example, $E_3 = \{(1,1,1),(3)\}$ and $O_3 = \{(2,1),(1,2)\}$.]

Use your bijection to conclude how many compositions there are with an even number of even parts.

(d) Show that the total number of all parts in all compositions of n is $(n+1)2^{n-2}$.
[For example, the compositions of 3 are $(3),(2,1),(1,2)$, and $(1,1,1)$, which all together have 8 parts; and $8 = (3+1)2^{3-2}$.]
[Hint: Use a stars and bars argument: if you line up all the compositions, how many bars appear in total? Explain why the total number of parts is equal to the total number of bars, plus the total number of compositions.]

Exercise 10.

(a) Explain why there are $\binom{n-1}{\ell}$ positive integer solutions to

$$x_1 + x_2 + \cdots + x_\ell < n$$

and $\binom{n+\ell}{\ell}$ non-negative integer solutions to

$$x_1 + x_2 + \cdots + x_\ell \leq n$$

by setting up a linear equations that have the appropriate number of solutions.

(b) (i) How many solutions are there to the equation

$$x_1 + x_2 + x_3 = 10,$$

where x_1, x_2, and x_3 are nonnegative integers? How many solutions are there if x_1, x_2, and x_3 are positive integers?

(ii) How many solutions are there to the equation

$$x_1 + x_2 + x_3 \leq 10,$$

where x_1, x_2, and x_3 are nonnegative integers?

Exercise 11.

(a) Suppose you've got eight varieties of doughnuts to choose from at a doughnuts shop.

  (i) How many ways can you pick 6 doughnuts?

  (ii) How many ways can you pick a dozen doughnuts?

  (iii) How many ways can you pick a dozen doughnuts with at least one of each kind?

(b) How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a jar contain if it has 20 coins in it?

Exercise 12.

(a) List the 3-multisets on [2].

(b) How many 5-multisets are there on [8]?

(c) How many multisets are there on [8]? (Of any size)

(d) Describe a bijection between k-multisets on $[n]$ and stars and bars arrangements with k stars and $n-1$ bars.

Exercise 13.

(a) Prove

$$(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots(1+x_n+x_n^2+\cdots)=\sum_{M=(S,\nu)}\prod_{x_i\in S}x_i^{\nu(x_i)},$$

by induction on n.

(b) Show algebraically that $\binom{-n}{k}(-1)^k=\binom{n+k-1}{k}$.

(c) (a) Write the generating function (both series and closed form) for the number of weak compositions of n with k parts.
[Hint: This should look something like the generating function for multisets.]

  (b) Write the generating function (both series and closed form) for the number of (not weak) compositions of n with k parts.

  (c) Write the generating function for the number of weak compositions of n with k parts, all less than j.

  (d) Item give a generating function proof that the number of weak compositions of n into k parts, with each part less than j, is

$$\sum_{\substack{r,s\in\mathbb{N}\\ r+sj=n}}(-1)^s\binom{k+r-1}{r}\binom{k}{s}.$$