Combinatorial Analysis – 9/8/15

Warmup. Recall the binomial theorem says

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

We showed on HW1 that

$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} \qquad (*)$$

by a combinatorial proof (*"Let A and B be disjoint sets satisfying $|A| = a$ and $|B| = b$, and count the size-n subsets of $A \sqcup B$ in two ways..."*).
Now use the binomial theorem and the multiplication rule

$$\left(\sum_i a_i x^i\right)\left(\sum_j b_j x^j\right) = \sum_k \left(\sum_{\ell=0}^{k} a_\ell b_{k-\ell}\right) x^k$$

to give a generating function proof of $(*)$.

Exercise 7. Use $\frac{1}{1-x} = \sum_{n\in\mathbb{N}} x^n$ and $e^x = \sum_{n\in\mathbb{N}} \frac{x^n}{n!}$ as *definitions* of $\frac{1}{1-x}$ and e^x, i.e.

$$\frac{1}{1-e^x} \text{ is short-hand for } F(G(x)), \text{ where } \begin{array}{l} F(x) = \sum_{n\in\mathbb{N}} x^n, \text{ and} \\ G(x) = \sum_{n\in\mathbb{N}} \frac{x^n}{n!}. \end{array}$$

Which of the following expressions are well-defined formal power series? Why? For those expressions that are well-defined, give their first few terms.

(i) e^{x+1} (ii) e^{x+3x^2} (iii) e^{e^x} (iv) e^{e^x-1} (v) $\dfrac{1}{1-xe^x}$ (vi) $\dfrac{1}{xe^x}$

Exercise 8. (EC1, exercise 1.8)

(a) Use the generalized binomial theorem to expand $\frac{1}{\sqrt{1-4x}}$ in series form.

(b) Calculate $\frac{(2n)!}{n!}$ for $n = 1, 2, 3$. What is $\frac{(2n)!}{n!}$ in general?

(c) Calculate $\binom{1/2}{k}$ for $k = 1, 2, 3$ What is $\binom{1/2}{k}$ in general? [Note that you can factor $\frac{1}{2}$ from every term in the numerator. Then use part (b).]

(d) Conclude

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

(e) Give a combinatorial proof of the identity $2\binom{2n-1}{n} = \binom{2n}{n}$.

(f) Find $\sum_{n=0}^{\infty} \binom{2n-1}{n} x^n$.