Combinatorial Analysis – 9/3/15

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \qquad \text{and} \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Warmup. Last time, we showed

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\sum_{\ell=0}^{\infty} x^\ell = \sum_{n=0}^{\infty}(n+1)x^n. \tag{$*$}$$

Working from the lowest degree terms, verify ($*$) by expanding the first few terms of

$$\frac{1}{(1-x)^2} = (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots).$$

In other words, the constant term is $1 * 1 = 1$; the degree 1 term is $1 * x + x * 1 = 2x$; and so on. Similarly, we can use substitution to show

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} 2^n \frac{x^n}{n!}. \tag{$**$}$$

Verify ($**$) by expanding the first few terms of

$$e^{2x} = (e^x)^2 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right).$$

Exercise 4. (a) Give both the generating and exponential generating functions for

$$f(n) = 3^n; \qquad g(n) = 3; \qquad \varphi(n) = 3n; \qquad \text{and} \qquad \psi(n) = n!3^n; \qquad \text{for } n \in \mathbb{N}.$$

For each, give your answer in series form. Whenever possible, also give your answer in closed form.

(b) Verify the rule for multiplying basic and exponential formal series for $I = \mathbb{N}$, for the first 4 coefficients. In other words, calculate c_n for $n = 0, 1, 2, 3$ by multiplying out the left hand side of

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots,$$

and comparing coefficients (and similarly for the exponential case).

(c) Verify that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

by solving for b_n in the equation

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n,$$

where

$$a_0 = a_1 = c_0 = 1, \text{ and } a_n, c_n = 0 \text{ otherwise},$$

i.e., $\sum_{n=0}^{\infty} a_n x^n = 1 + x$ and $\sum_{n=0}^{\infty} c_n x^n = 0$. [See EC1, Example 1.1.5; but be more explicit.]

Exercise 5. For each of the following identities,

(i) check by hand for $n = 3$;
(ii) verify using the binomial theorem, evaluating for specific values of x;
(iii) give a combinatorial proof of the identity.

(a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$;

(b) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ for $n > 0$

[Hint: for (ii), differentiate first].

(c) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ for $n > 0$

[Hint: For (iii), rewrite the identity by moving the negative terms to the right (generally good practice for combinatorial proofs). Construct a bijection between the set E_n of all subsets of $[n]$ that have an even number of elements and O_n, the odd counterpart.].

Exercise 6. Use the multiplication rules for exponential series to show that

$$1 = e^x e^{-x} \qquad \text{implies} \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \quad \text{for } n > 0$$

(our third proof, making EC1, Example 1.1.6 more explicit).

Exercise 7. Use $\frac{1}{1-x} = \sum_{n\in\mathbb{N}} x^n$ and $e^x = \sum_{n\in\mathbb{N}} \frac{x^n}{n!}$ as *definitions* of $\frac{1}{1-x}$ and e^x, i.e.

$$\frac{1}{1-e^x} \text{ is short-hand for } F(G(x)), \text{ where } \begin{array}{l} F(x) = \sum_{n\in\mathbb{N}} x^n, \text{ and} \\ G(x) = \sum_{n\in\mathbb{N}} \frac{x^n}{n!}. \end{array}$$

Which of the following expressions are well-defined formal power series? Why? For those expressions that are well-defined, give their first few terms.

$$\text{(i) } e^{x+1} \qquad \text{(ii) } e^{x+3x^2} \qquad \text{(iii) } e^{e^x} \qquad \text{(iv) } e^{e^x-1} \qquad \text{(v) } \frac{1}{1-xe^x} \qquad \text{(vi) } \frac{1}{xe^x}$$