Combinatorial Analysis – 9/1/15

Exercise 3. (From last time)

(a) Explain why $\binom{n}{k} = \frac{(n)_k}{k!}$ directly using product and division rules.

(b) Give a combinatorial proof of the identity

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n},$$

where $a, b, n \in \mathbb{N}$ and $a, b \geq n$. [Hint: Consider two disjoint sets A and B, with $|A| = a$ and $|B| = b$. How many subsets does $A \sqcup B$ have?]

Recall the Taylor series expansions

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \qquad \text{and} \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Warmup. Use substitution (function composition), differentiation, and integration to give the series expansions for the following functions, for $c \in \mathbb{R}$, $c \neq 0$.

$$\frac{1}{1-cx}, \qquad \frac{1}{1-x^c}, \qquad e^{cx}, \qquad \frac{1}{(1-x)^2}, \qquad \frac{1}{(1-x)^3}, \qquad \ln|1-x|.$$

[For example, $\frac{d}{dx}\frac{1}{1-x} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$, so the series expansion of $\frac{1}{(1-x)^2}$ is the derivative of the series expansion of $\frac{1}{1-x}$]

Exercise 4.

(a) Give both the generating and exponential generating functions for

$$f(n) = 3^n; \qquad g(n) = 3; \qquad \varphi(n) = 3n; \qquad \text{and} \qquad \psi(n) = n!3^n; \qquad \text{for } n \in \mathbb{N}.$$

For each, give your answer in series form. Whenever possible, also give your answer in closed form.

(b) Verify the rule for multiplying basic and exponential formal series for $I = \mathbb{N}$, for the first 4 coefficients. In other words, calculate c_n for $n = 0, 1, 2, 3$ by multiplying out the left hand side of

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots,$$

and comparing coefficients (and similarly for the exponential case).

(c) Verify that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

by solving for b_n in the equation

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n,$$

where

$$a_0 = a_1 = c_0 = 1, \text{ and } a_n, c_n = 0 \text{ otherwise},$$

i.e., $\sum_{n=0}^{\infty} a_n x^n = 1 + x$ and $\sum_{n=0}^{\infty} c_n x^n = 0$. [See EC1, Example 1.1.5; but be more explicit.]

Exercise 5. For each of the following identities,

(i) check by hand for $n = 3$;
(ii) verify using the binomial theorem, evaluating for specific values of x;
(iii) give a combinatorial proof of the identity.

(a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n;$

(b) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$
[Hint: for (ii), differentiate first].

(c) $\displaystyle\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$
[Hint: For (iii), rewrite the identity by moving the negative terms to the right (generally good practice for combinatorial proofs). Construct a bijection between the set E_n of all subsets of $[n]$ that have an even number of elements and O_n, the odd counterpart.].