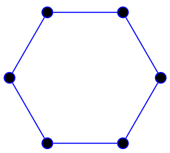
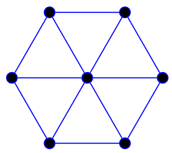
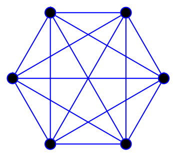
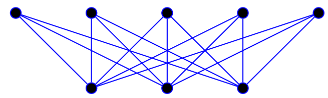
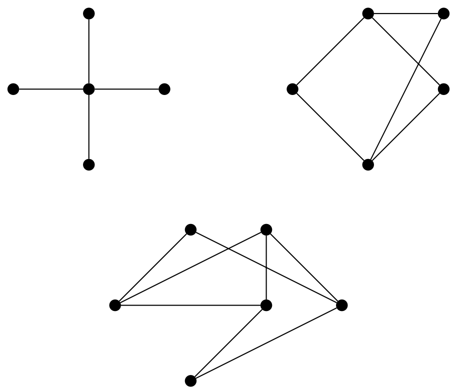
Exercise 28. (a) Draw C_6, W_6 K_6, and $K_{5,3}$.

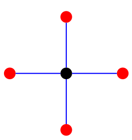
Solution:

C_6 :  W_6 :  K_6 :  $K_{5,3}$: 

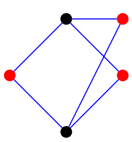
(b) Which of the following are bipartite? Justify your answer.



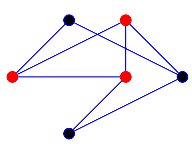
Solution:



Bipartite: put the red vertices in V_1 and the black in V_2.



Bipartite: put the red vertices in V_1 and the black in V_2.

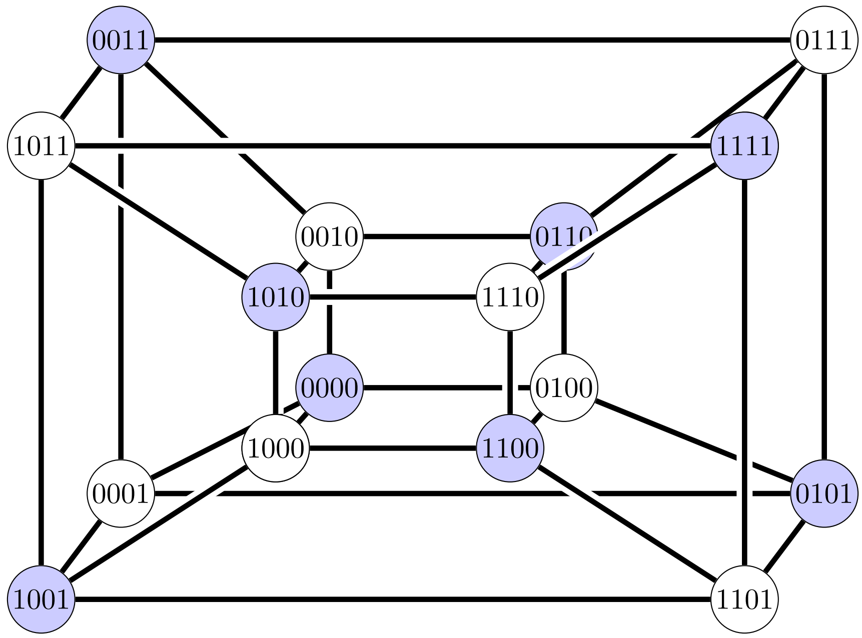


Not bipartite!

Consider the three vertices colored red. For the sake of contradiction, assume that it is bipartite. Pick any one of them to be in V_1. That would force the other two to be in V_2. But they are adjacent, which is a contradiction.

(c) Hypercubes are bipartite.

(i) The following is the 4-cube:



Shade in the vertices that have an even number of 0's. Explain why the 4-cube is bipartite.

Solution: None of the shaded vertices are pairwise adjacent. None of the non-shaded vertices are pairwise adjacent. So put all the shaded vertices in V_1 and all the rest in V_2 to see that Q_4 is bipartite.

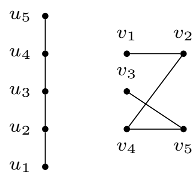
(ii) Explain why Q_n is bipartite in general.
[Hint: consider the parity of the number of 0's in the label of a vertex.]

Solution: Any two vertices with an even number of 0's differ in at least two bits, and so are non-adjacent. Similarly, any two vertices with an odd number of 0's differ in at least two bits, and so are non-adjacent. So let $V_1 = \{$ vertices with an even number of 0's $\}$ and $V_2 = \{$ vertices with an odd number of 0's $\}$.

Exercise 29.

(a) For each of the following pairs, list their degree sequences. Then are they isomorphic?. If not, why? If yes, give an isomorphism.

(i)



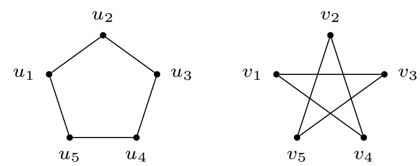
deg seq: 2,2,2,1,1

Isomorphic:

$u_1 \mapsto v_1,\ u_2 \mapsto v_2,\ u_3 \mapsto v_4,$

$u_4 \mapsto v_5,\ u_5 \mapsto v_3$

(ii)



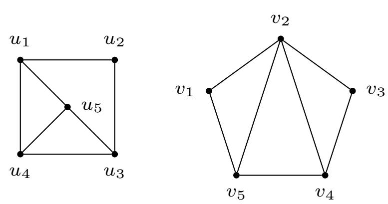
deg seq: 2,2,2,2,2

Isomorphic:

$u_1 \mapsto v_1,\ u_2 \mapsto v_3,\ u_3 \mapsto v_5$

$u_4 \mapsto v_2,\ u_5 \mapsto 4.$

(iii)

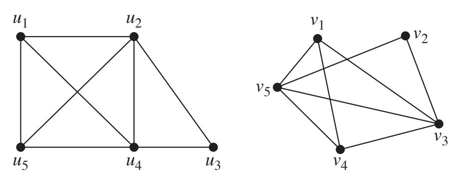


Left deg seq: 3,3,3,3,2;

Right deg seq: 4,3,3,2,2.

Not isomorphic: different deg seq's.

(iv)



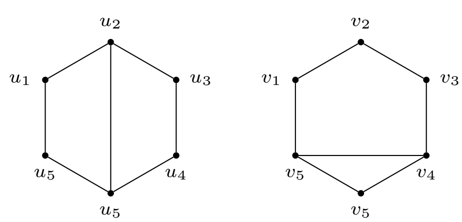
Deg seq: 4,4,3,3,2

Isomorphic:

$u_3 \mapsto v_2$, $u_2 \mapsto v_3$, $u_4 \mapsto v_5$

$u_1 \mapsto v_1$, $u_5 \mapsto v_4$

(v)



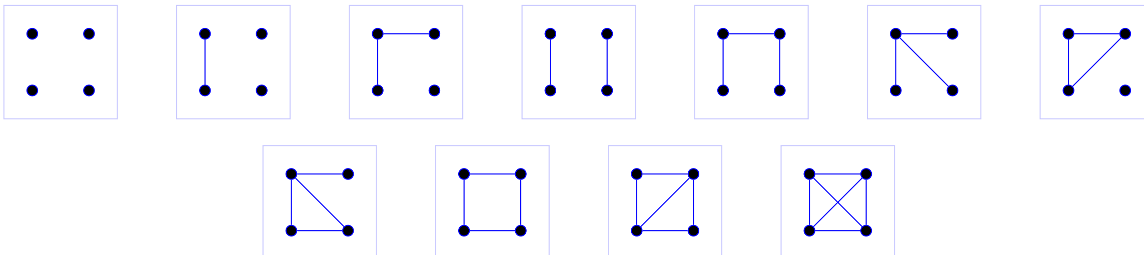
Deg seq: 3,3,2,2,2,2

Not isomorphic:

Right has a 3-cycle; Left doesn't.

(b) How many isomorphism classes are there for simple graphs with 4 vertices? Draw them.

Solution:

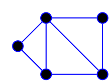


(c) How many edges does a graph have if its degree sequence is $4, 3, 3, 2, 2$? Draw a graph with this degree sequence. Can you draw a simple graph with this sequence?

Solution: By the handshake lemma,

$$2|E| = 4 + 3 + 3 + 2 + 2 = 14.$$

So there are 7 edges. Here is an isomorphism class of simple graphs that has that degree sequence:



(d) For which values of n, m are these graphs regular? What is the degree?

(i) K_n (ii) C_n (iii) W_n (iv) Q_n (v) $K_{m,n}$

Solution:

(i) K_n: Regular for all n, of degree $n - 1$.
(ii) C_n: Regular for all n, of degree 2.
(iii) W_n: Regular only for $n = 3$, of degree 3.
(iv) Q_n: Regular for all n, of degree n.
(v) $K_{m,n}$: Regular for $n = m$, n.

(e) How many vertices does a regular graph of degree four with 10 edges have?

Solution: By the handshake theorem,

$$2 * 10 = |V| * 4$$

so $|V| = 5$.

(f) Show that every non-increasing finite sequence of nonnegative integers whose terms sum to an even number is the degree sequence of a graph (where loops are allowed). Illustrate your proof on the degree sequence 7,7,6,4,3,2,2,1,0,0. [Hint: Add loops first.]

Solution: For a degree sequence $d_1, d_2, \ldots, d_n$, draw one vertex v_i for each degree d_i, and attach $\lfloor d_i/2 \rfloor$ loops attached to v_i. Then for each i for which d_i is even, v_i so far had degree d_i. For the i for which d_i is odd, v_i currently has degree $d_i - 1$. Since the terms sum to an even number, there must be an even number of i for which d_i is odd; pair these i's up: the first with the second, the third with the fourth, and so on. Now draw an edge between all such paired vertices. The resulting graph has the appropriate degree sequence.

(g) Show that isomorphism of simple graphs is an equivalence relation.

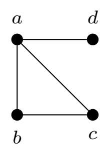
Solution:

(a) Reflexive: the identity map on vertices is an isomorphism of a graph to itself.

(b) Symmetric: If f is an isomorphism $f : G_1 \to G_2$, then $f : V_1 \to V_2$ is bijective, and therefore has an inverse. Since f preserves adjacency, so does f^{-1}. So $f^{-1} : G_2 \to G_1$ is an isomorphism.

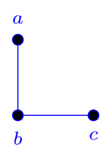
(c) Transitive: If $f : G_1 \to G_2$ and $g : G_2 \to G_3$ are isomorphisms, then $g \circ f : G_1 \to G_3$ is an isomorphism, since the composition of bijective and edge-preserving maps is bijective and edge-preserving.

Exercise 30. (a) Consider the graph

$G =$ 

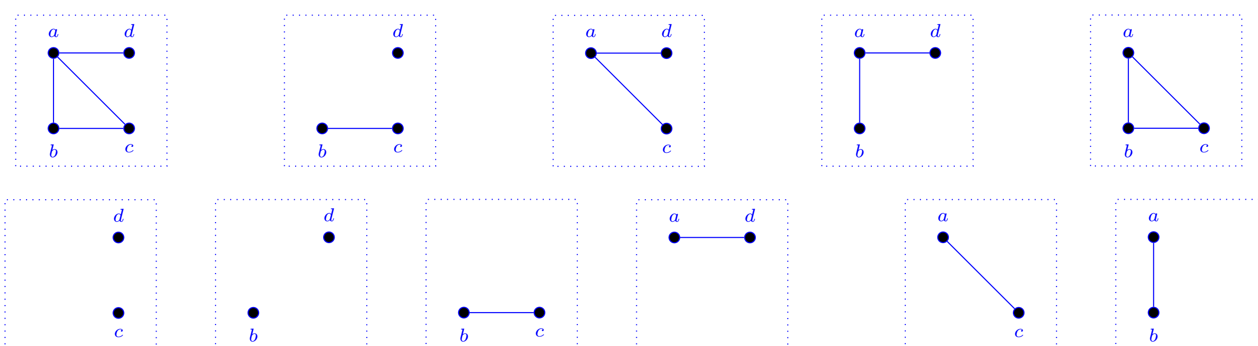
(i) Give an example of a subgraph of G that is not induced.

Solution:

$H =$ 

(ii) How many induced subgraphs does G have? List them.

Solution: There are 4 vertices, so there are 2^4 induced subgraphs:



 and $\emptyset$

(iii) How many subgraphs does G have?

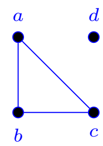
Solution: A graph with m edges has exactly 2^m subgraphs with the same vertex set. So, going through the induced subgraphs (the largest subgraph of G with each possible vertex set), we get

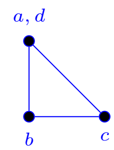
$$2^4 + 2 + 2^2 + 2^2 + 2^3 + 1 + 1 + 2 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1$$

subgraphs of G in total.

(iv) Let e be the edge connecting a and d. Draw $G - e$ and G/e.

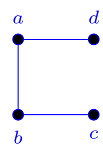
Solution:

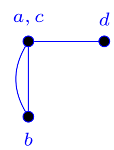
$G - e =$ 

$G/e =$ 

(v) Let e be the edge connecting a and c. Draw $G - e$ and G/e.

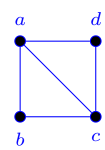
Solution:

$G - e =$ 

$G/e =$ 

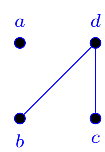
(vi) Let e be an edge connecting d and c. Draw $G + e$.

Solution:

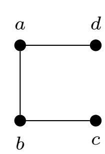
$G =$ 

(vii) Draw $\bar{G}$.

Solution:

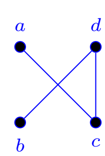
$G =$ 

(b) Show that

$G =$ 

is isomorphic to its complement.

Solution: Since

$\bar{G} =$ 

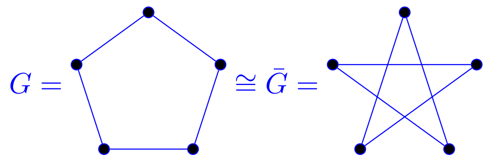
the map

$$d \mapsto a, \qquad a \mapsto c, \qquad b \mapsto d, \qquad c \mapsto b,$$

gives an isomorphism.

(c) Find a simple graph with 5 vertices that is isomorphic to its own complement. (Start with: how many edges must it have?)

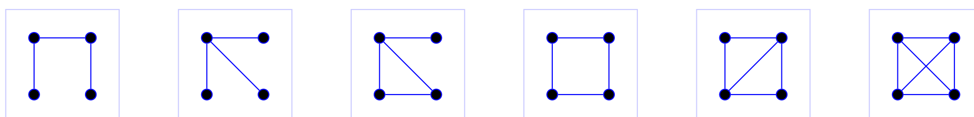
Solution: Since there are 10 possible edges, G must have 5 edges. One example that will work is C_5:



Exercise 31.

(a) Draw the isomorphism classes of connected graphs on 4 vertices, and give the vertex and edge connectivity number for each.

Solution:



$\kappa = 1$	$\kappa = 1$	$\kappa = 1$	$\kappa = 2$	$\kappa = 2$	$\kappa = 3$
$\lambda = 1$	$\lambda = 1$	$\lambda = 1$	$\lambda = 2$	$\lambda = 2$	$\lambda = 3$

(b) Show that if v is a vertex of odd degree, then there is a path from v to another vertex of odd degree.

Solution: By the Handshake Theorem, every graph has an even number of odd degree vertices. Notice that each connected component is an induced subgraph with the same degrees. So each connected component also has an even number of odd degree vertices. So if a connected component has an odd degree vertex, it must have two. So those two vertices are connected by a walk.

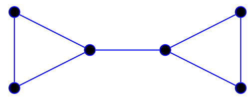
(c) Prove that for every simple graph, either G is connected, or $\bar{G}$ is connected.

Solution: Suppose G is not connected. Let $H_1, H_2, \ldots, H_k$ be the connected components of G (i.e. the subgraphs induced by each set of vertices determined by the connected components). First, consider two vertices in different connected components in G: $u \in H_i$, $v \in H_j$, $i \neq j$. Since u and v are in different connected components in G, they are certainly not adjacent; thus u and v are adjacent in $\bar{G}$, and therefore in the same connected component. Now consider two vertices in the same connected component in G: $u, v \in H_i$. Since there is more than one connected component in G, let $w \in H_j$, $i \neq j$. By our previous argument, u and v are both in the neighborhood of w in $\bar{G}$, and so u, w, v is a path in $\bar{(G)}$. Thus u and v are connected in $\bar{G}$. Thus $\bar{G}$ is connected.

(d) Recall that $\kappa(G)$ is the vertex connectivity of G and $\lambda(G)$ is the edge connectivity of G. Give examples of graphs for which each of the following are satisfied.

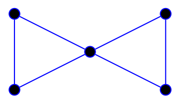
Let $\delta = \min_{v \in V} \deg(v)$.

(i) $\kappa(G) = \lambda(G) < \min_{v \in V} \deg(v)$



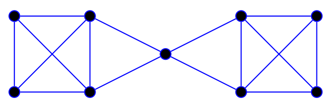
$\kappa = \lambda = 1, \delta = 2.$

(ii) $\kappa(G) < \lambda(G) = \min_{v \in V} \deg(v)$



$\kappa = 1, \lambda = 2, \delta = 2.$

(iii) $\kappa(G) < \lambda(G) < \min_{v \in V} \deg(v)$



$\kappa = 1, \lambda = 2, \delta = 3.$

(iv) $\kappa(G) = \lambda(G) = \min_{v \in V} \deg(v)$

Cycles of length 3 or more have $\kappa = \lambda = \delta = 2$.

(e) For the following theorem, pick any of parts (ii)–(iv) and show (carefully!) that it's equivalent to part (i).

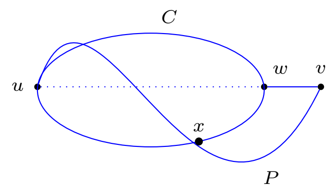
Theorem: For a simple graph with at least 3 vertices, the following are equivalent.

(i) G is connected and contains no cut vertex.
(ii) Every two vertices in V are contained in some cycle.
(iii) Every two edges in E are contained in some cycle, and G contains no isolated vertices.
(iv) For any three vertices $u, v, w \in V$, there is a path from u to v containing w.

Solution: (ii) $\implies$ (i): If every two vertices u and v are contained in some cycle, then there are two internally disjoint $u-v$ paths. Thus u and v are connectes, and the deletion of any third vertex will not disconnect u from v. Thus $\kappa(G) \geq 2$.

(i) $\implies$ (ii): Assume $\kappa(G) \geq 2$. Consider $u, v \in V$, and set $d = d(u, v)$.
If $d = 1$, then u and v are adjacent. Since $|V| \geq 3$, there is some $w \in V$ distinct from u and v. Since $\kappa(G) \geq 2$, there is some $u - w$ path P_1 not containing v; similarly there is some $w - v$ path P_2 not containing u. Let x be the first vertex on P_1 which is also on P_2 (since w is on both paths, such a vertex exists). Then the walk from u on P_1 to x, then on P_2 to v, and finally back on the edge vu, is a cycle containing u and v.
If $d > 1$, suppose that for any $w \in V$ with $d(u, z) < d$, there is some cycle containing u and w. Consider a minimal length $u-v$ path, and let w be v's neighbor on this path, so that $d(u, w) = d-1$. By our inductive hypothesis, there is a cycle C containing u and w. Either C also contains v, in which we also have a cycle containing u and v, or it doesn't. If C does not contain v, then since $\kappa(G) \geq 2$, we also have a path P from v to u not containing w. Let x be the first vertex on P

which is also on C (since u is on P and C, x exists).



Then the walk from v along P to x, then along C away from w and toward u (if $x = u$, walk in either direction), through u and around to w, and finally along the edge wv, is a cycle containing u and v.

(iii) $\implies$ (i): Consider $u, v \in V$. Since there are no isolated vertices in G, $\deg(u), \deg(v) \geq 1$, so u and v are each incident to at least one edge. Since $|V| \geq 3$, there are then at least two edges. Since every pair of edges are contained in a common cycle, a vertex incident to an edge must be incident to at least two edges. Thus there are distinct edges $e \neq f$ with e incident to u and f incident to v. Since e and f are contained in some cycle, u and v are contained in that same cycle. So there are two internally disjoint $u - v$ paths. Thus u and v are connected, and the deletion of any third vertex will not disconnect u from v. Thus $\kappa(G) \geq 2$.

(i) $\implies$ (iii): With $G = (V, E, \phi)$ and $e, f \in E$, define

$$d = d(e, f) = \min_{\substack{u \in \phi(e) \\ v \in \phi(f)}} d(u, v).$$

Now assume $\kappa(G) \geq 2$. Fix $e, f \in E$ distinct and let $d = d(e, f)$. Choose $u \in \phi(e)$ and $v \in \phi(f)$ with $d(u, v) = d$, and let u' and v' be the other vertices incident to e and f, respectively.
First suppose $d = 0$ (so that e and f are incident to a common vertex). Since $\kappa(G) \geq 2$, there is a path P from u' to v' not containing $u = v$. Then the walk from u' along P to v', then back along f then e is a cycle containing e and f.
If $d > 0$, then assume that for all $e \neq g \in E$ for which $d(e, g) < d$ is contained in a cycle together with e. Take a minimal-length path from u to v, and let w be v's neighbor on P, so that $d(u, w) = d - 1$, and so $d(e, wv) = d - 1$. By our inductive hypothesis, there is a cycle C containing e and wv. If v' is on C, then the walk along $v'v$, then along C away from v' and toward u, then on around to v', is a cycle containing e and $v'v$. Otherwise, let P be a path from v' to u not containing v (which exists since $\kappa(G) \geq 2$). Let x be the first vertex on P which is also on C. Then the walk from v along v', then along P to x, then along C away from v and toward u, through on to v, is a cycle containing e and f.

(iv) $\implies$ (i): Let $u, v \in V$. For any $w \in V$ distinct from u and v, assuming (iv) gives at least one $u - w$ path containing v. The initial walk from u to v on this path gives a $u - v$ path not containing w. So u and v are connected in G and in $G - w$. Thus $\kappa(G) \geq 2$.

(i) $\implies$ (iv): Assume $\kappa(G) \geq 2$, let $u, v, w \in V$, and let $d = d(u, w)$.

If $d = 1$, let P be a path from w to v not containing u (which exists because $\kappa(G) \geq 2$). Then the walk from u to w and then along P to v is a path from u to v containing w.
If $d > 1$, assume that for every $x \in V$ with $d(u, x) < d$, there is a $u - v$ path through x. Take a minimal-length path from u to w, and let x be w's neighbor on this path, so that $d(u, x) = d - 1$. Then our inductive hypothesis guarantees a $u - v$ path P containing x. If P also contains w, then P is a $u - v$ path containing w as desired. Otherwise, let P' be a path from w to u not containing x. Let z be the first vertex on P' which is also on P (which is guaranteed since u is on both P and P'). If z sits between x and v on P, then the walk from u along P to x, then along xw, then along P' to z, then continuing along P to v, is a $u - v$ path containing w. Otherwise, z sits between u and x on P, so that the walk from u along P to z, the backwards along P' toward w, up wx, and then along P toward v is a $u - v$ path through w.