Exercise 25. (a) Write the permutation matrices for each $w \in \mathcal{S}_3$.

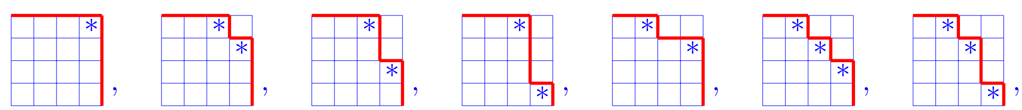
Solution:

$$P(123) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P(132) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P(213) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

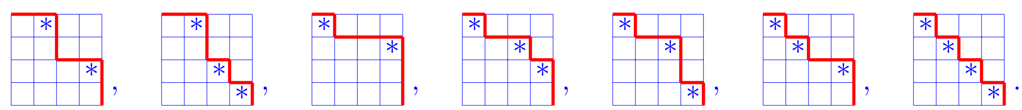
$$P(231) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad P(312) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P(321) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(b) Verify that there are 14 of permutations of 4 that have no decreasing sequences of length 3 by counting the corresponding lattice paths, and listing the corresponding permutations.

Solution:



$w = 4123$, $w = 3412$, $w = 3142$, $w = 3124$, $w = 2413$, $w = 2341$, $w = 2314$



$w = 2143$, $w = 2134$, $w = 1423$, $w = 1342$, $w = 1324$, $w = 1243$, $w = 1234$

(c) For each of the following pairs (u, w), decide whether or not w is u-avoiding.

(i) $u = 132$, $w = 7421365$; No: $7\underline{4}213\underline{65}$

(ii) $u = 132$, $w = 5671234$; Yes.

(iii) $u = 1234$, $w = 1765423$; Yes.

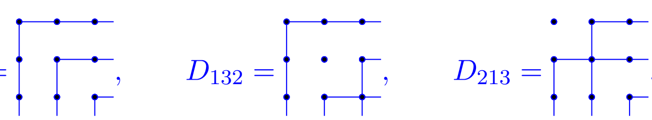
(iv) $u = 1234$, $w = 73164258$; No: $w = 73\underline{1}64\underline{258}$

(v) $u = 1234$, $w = 123$. Yes: w is not long enough to admit u.

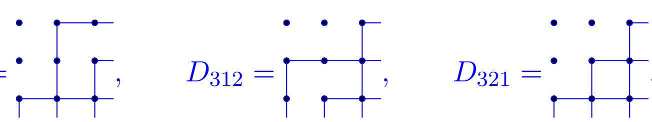
(d) Draw the dot diagrams for each $w \in \mathcal{S}_3$.

Solution:

$D_{123} =$, $D_{132} =$, $D_{213} =$,



$D_{231} =$, $D_{312} =$, $D_{321} =$.



(e) Inverses of dot diagrams.

(i) Verify that $D_w^t = D_{w^{-1}}$ for $w \in \mathcal{S}_3$.

Solution: For each every $w \in \mathcal{S}_3$ other than 312 and 231, $w = w^{-1}$, and the corresponding dot diagram is symmetric across the diagonal. For the remaining two permutations, they are inverses of each other, and their corresponding diagrams are each others' reflection across the diagonal.

(ii) Explain why $D_w^t = D_{w^{-1}}$.

Solution: If $w(i) = j$, then in D_w, every dot in column j and in rows greater than or equal to i are crossed out; and every dot in row i and in columns greater than or equal to j are crossed out. Also, $w^{-1}(j) = i$, so the diagram $D_{w^{-1}}$ has the same dots crossed out, except with rows and columns exchanged. This is exactly the transpose of D_w, as desired.

(f) Inversion tables from dot diagrams.

(i) Verify that the inversion tables for the permutations $w \in \mathcal{S}_3$ can be read off of the dot diagrams D_w.

Solution: In all diagrams, there are no dots in the third column, and for all permutations, $a_3 = 0$. For a_1 and a_2:

w:	dots in col 1:	{inv where 1 loses}:	dots in col 2:	inv where 2 loses:
123	0	$\emptyset$	0	$\emptyset$
132	0	$\emptyset$	1	$\{(3,2)\}$
213	1	$\{(2,1)\}$	0	$\emptyset$
231	2	$\{(2,1),(3,1)\}$	0	$\emptyset$
312	1	$\{(3,1)\}$	1	$\{(3,2)\}$
321	2	$\{(3,1),(2,1)\}$	1	$\{(3,2)\}$

(ii) Explain why the inversion table $I = (a_1, \ldots, a_n)$ corresponding to a permutation w is given by

$$a_i = \#\{ \text{ circled dots in column } i \text{ of } D_w \ \}.$$

Solution: If the dot (i,j) is in the diagram D_w, then column j is crossed out below row i (meaning $w(\ell) = j$ for $\ell > i$) and row i is crossed out after column j (meaning $w(i) > j$). So $w(\ell) = j < w(i)$ and $\ell > i$, so that $(w(i), j)$ is an inversion in w (where j loses). Similarly, if $(w(i), j)$ is an inversion in w, then the dot (i,j) is in D_w. Therefore, the dots in column j of D_w are in bijection with the inversions in w where j loses.

(g) Verify that the 132-avoiding permutations of [3] are Ferrers diagrams, and in bijection with the set of partitions of $0, 1, \ldots, \binom{3}{2}$ satisfying $\lambda_i \le n - i$.

Solution:

D_{123} is the Ferrers diagram for the partition $\emptyset$;
D_{132} is not 132-avoiding;
D_{213} is the Ferrers diagram for the partition (1);
D_{231} is the Ferrers diagram for the partition $(1,1)$;
D_{312} is the Ferrers diagram for the partition (2);
D_{321} is the Ferrers diagram for the partition $(2,1)$.

(h) Verify Proposition 1.5.1, i.e. that

$$\sum_{w \in \mathcal{S}_{132}(n)} q^{\mathrm{inv}(w)} = \sum_{\lambda} q^{|\lambda|},$$

for $n = 3$.

Solution:

$$\sum_{w\in\mathcal{S}_{132}(n)} q^{\mathrm{inv}(w)} = q^{\mathrm{inv}(123)} + q^{\mathrm{inv}(213)} + q^{\mathrm{inv}(231)} + q^{\mathrm{inv}(312)} + q^{\mathrm{inv}(321)}$$
$$= q^0 + q^1 + q^1 + q^2 + q^3 = 1 + 2q + q^2 + q^3$$
$$\sum_{\lambda} q^{|\lambda|} = q^{|\emptyset|} + q^{|(1)|} + q^{|(1,1)|} + q^{|(2)|} + q^{|(2,1)|}$$
$$= q^0 + q^1 + q^1 + q^2 + q^3 = 1 + 2q + q^2 + q^3$$

Exercise 26. Go to

http://www-math.mit.edu/∼rstan/ec/catalan.pdf

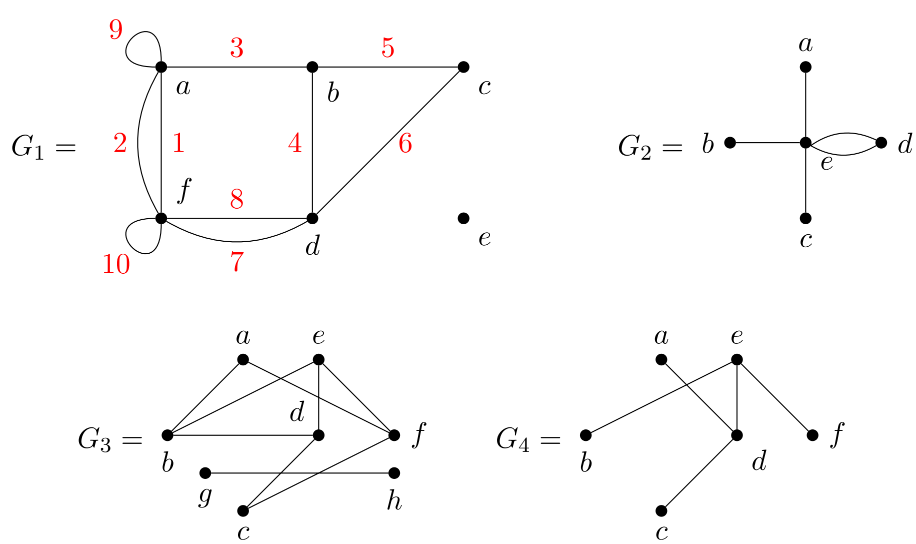
and

http://www-math.mit.edu/∼rstan/ec/catadd.pdf

to see the famous Catalan numbers problem (6.19 in EC2). Skim the whole thing. Pick three unrelated parts and verify that the corresponding combinatorial set is of size 14 for $n = 4$. Unrelated means that if you do one part concerning lattice paths, then the other parts you pick should not concern lattice paths. Parts (h) and (i) are equivalent to the lattice paths you did in problem 25(b), so don't repeat these. You may need to look up or ask for definitions, but examples for $n = 3$ are almost always given.

Solution: Just do it.

Exercise 27. Let



(a) For each of the graphs G_1, G_2, G_3, and G_4, answer the following questions.
   (i) Is G_i simple? Is G_i a multigraph? Is G_i connected? Is G_i a forrest? Is G_i a tree?
   (ii) What is the neighborhood of a? of e? of $\{a, e\}$?
   (iii) What is the degree of a? of e?
   (iv) Verify the handshake theorem.
   (v) Verify that there are an even number of vertices of odd degree.

Solution: G_1:

(i) The graph G_1 is not simple nor connected, nor is it a multigraph, a forrest, or a tree.
(ii) $N(a) = \{a, b, f\}$, $N(e) = \emptyset$, and $N(\{a, e\}) = \{a, b, f\}$.
(iii) $\deg(a) = 5$, $\deg(e) = 0$.
(iv) $\sum_{v \in V} \deg(v) = 5 + 3 + 2 + 4 + 6 + 0 = 20 = 2(10) = 2|E| \checkmark$.
(v) The vertices of odd degree are $\{a, b\}$.

G_2:

(i) The graph G_2 is not simple, nor is it a a forrest, or a tree. It is connected and it is a multigraph.
(ii) $N(a) = \{e\}$, $N(e) = \{a, b, c, d\}$, and $N(\{a, e\}) = V$.
(iii) $\deg(a) = 1$, $\deg(e) = 5$.
(iv) $\sum_{v \in V} \deg(v) = 1 + 1 + 1 + 2 + 5 = 10 = 2(5) = 2|E| \checkmark$.
(v) The vertices of odd degree are $\{a, b, c, e\}$.

G_3:

(i) The graph G_3 is not connected, nor is it a forrest, or a tree. It is simple, and therefore it is a multigraph.
(ii) $N(a) = \{b, f\}$, $N(e) = \{b, d, f\}$, and $N(\{a, e\}) = \{b, d, f\}$.
(iii) $\deg(a) = 2$, $\deg(e) = 3$.
(iv) $\sum_{v \in V} \deg(v) = 2 + 3 + 2 + 3 + 3 + 3 + 1 + 1 = 18 = 2(9) = 2|E| \checkmark$.
(v) The vertices of odd degree are $\{b, d, e, f, g, h\}$.

G_4:

(i) The graph G_4 is a tree, and therefore it is all of the above.
(ii) $N(a) = \{d\}$, $N(e) = \{b, d, f\}$, and $N(\{a, e\}) = \{b, d, f\}$.
(iii) $\deg(a) = 1$, $\deg(e) = 3$.
(iv) $\sum_{v \in V} \deg(v) = 1 + 1 + 1 + 3 + 3 + 1 = 10 = 2(5) = 2|E| \checkmark$.
(v) The vertices of odd degree are $\{a, b, c, d, e, f\}$.

(b) List three cycles in G_3.

$$(a, b, e, f, a), \qquad (a, b, d, e, f, a), \qquad (b, d, e, b)$$

(c) Give an example of a closed walk in G_1 that is not a cycle.

$$(b, 4, d, 4, b)$$

(d) Give an example of a path of length 3 in G_3.

$$(a, f, e, d)$$

(e) Give an example of a maximal path in G_4.

$$(b, e, f)$$

(f) Give a walk of length 5 in G_2 that is not closed, not a path, and not a trail.

$$(b, e, b, e, b, e)$$

(g) Give a trail of length 8 in G_1.

$$(a, 9, a, 2, f, 10, f, 1, a, 3, b, 5, c, 6, d, 7, f)$$

(h) What's the longest length of the trails in G_2? Ans: 4. For example, walk from b to e to d along on edge and back to e along the other, then to a.