Solutions for HW7

Exercise 24.

(a) Eulerian numbers.

(i) Verify that $A_3 = x + 4x^2 + x^3$ and $A_4 = x + 11x^2 + 11x^3 + x^4$.

Solution: Recall, $A(d,k)$ is the number of permutations of d with exactly $k-1$ descents. So for $\mathcal{S}_3$,

$$A(3,1) = |\{123\}| = 1,$$
$$A(3,2) = |\{132, 213, 231, 312, \}| = 4, \text{ and}$$
$$A(3,3) = |\{321\}| = 1.$$

So $A_3 = x + 4x^2 + x^3$. For $\mathcal{S}_4$,

$$A(4,1) = |\{1234\}| = 1 \qquad \text{and} \qquad A(4,4) = |\{4321\}| = 1.$$

For $A(4,2)$, we'll be a little more clever. If there is exactly one descent, it comes in position 1, 2, or 3. So since

$$|\{w \in \mathcal{S}_4 \mid D(w) = \{1\}\}| = |\{abcd \in \mathcal{S}_4 \mid a > b < c < d\} = 1$$
$$\text{(choose } a \neq 1\text{)},$$
$$|\{w \in \mathcal{S}_4 \mid D(w) = \{2\}\}| = |\{abcd \in \mathcal{S}_4 \mid a < b > c < d\} = \binom{4}{2} - 1 = 5$$
$$\text{(choose } \{a,b\} \neq \{1,2\}\text{), and}$$
$$|\{w \in \mathcal{S}_4 \mid D(w) = \{3\}\}| = |\{abcd \in \mathcal{S}_4 \mid a < b < c > d\} = 3$$
$$\text{(choose } d \neq 4\text{)}.$$

So $A(4,2) = 3 + 5 + 3$. Since all 24 permutations are accounted for in one of the $A(4,k)$, we have $A(4,3) = 24 - 1 - 11 - 1 = 11$. Thus $A_4 = x + 11x^2 + 11x^3 + x^4$.

(ii) In general, for $d > 0$, what are $A(d,1)$ and $A(d,d)$?

Solution:

$$A(d,1) = |\{123\cdots d\}| = 1 \qquad \text{and} \qquad A(d,d) = |\{d \cdots 321\}| = 1.$$

(iii) What is $A_d(1)$?

Solution:

$$A_d(1) = \sum_{w \in \mathcal{S}_d} 1^{1+\mathrm{des}(w)} = \sum_{w \in \mathcal{S}_d} 1 = |\mathcal{S}_d| = d!.$$

(b) Excedances.

(i) Show that $w = w_1 w_2 \cdots w_d$ has k weak excedances if and only if $u = u_1 u_2 \cdots u_d$, defined by $u_i = d + 1 - w_{d-i+1}$, has $d - k$ excedances.

Solution: If there is a weak excedance in position i, then $w_i \geq i$, so that

$$d + 1 - i \leq d + 1 - w_i = u_{d+1-i}$$

is not an excedance in u. On the other hand, if there is not a weak excedance in position i, then $w_i < i$, so that

$$d + 1 - i > d + 1 - w_i = u_{d+1-i}$$

is an excedance in u. Therefore if there are k weak excedances in w, there are $d - k$ excedances in u.

(ii) Show w has $d - 1 - j$ descents if and only if $w^{\mathrm{op}} = w_d w_{d-1} \cdots w_1$ has j descents.

Solution: If i is a descent in w, then $w_i > w_{i+1}$. So $d-i$ is not a descent in w^{op}. Similarly, if i is not a descent in w, then $d - i$ is a descent in w^{op}. Therefore if there are j descents in w, there are $(d - 1) - j$ descents in w^{op}.

(iii) Show that

$$A(d, k + 1) = |\{w \in \mathcal{S}_d \mid w \text{ has } k \text{ excedances}\}|$$

and

$$A(d, k + 1) = |\{w \in \mathcal{S}_d \mid w \text{ has } k + 1 \text{ weak excedances}\}|.$$

Solution: We have that

$$d - \mathrm{des}(\hat{w}) = |\{i \in [d] \mid w(i) \geq i\},$$

the number of weak excedances of w. So $\hat{w} \mapsto w$ gives a bijection between partitions $k+1$ weak excedances and $d - k - 1$ descents. By part bii, $w \mapsto w^{\mathrm{op}}$ gives a bijection between $d - k - 1$ descents and partitions with k descents. So,

$$\hat{w} \mapsto w \mapsto w^{\mathrm{op}}$$

gives a bijection between partitions $k+1$ weak excedances and partitions with k descents, giving the second desired identity.

In part bi, the map $u \mapsto w$ gives a bijection between partitions with k excedances and $d - k$ weak excedances, so that

$$u \mapsto w \mapsto \hat{w}$$

gives a bijection between partitions with k excedances and partitions with k descents, giving the first desired identity.

(c) Complete the proof of Prop. 1.4.6 by proving that $\mathrm{inv}(\gamma_k) = \mathrm{maj}(\eta_k)$ implies $\mathrm{inv}(\gamma_{k+1}) = \mathrm{maj}(\eta_{k+1})$ in the case where the last letter w_k of γ_k is smaller than w_{k+1}.

Solution: If the last letter w_k of γ_k is smaller than w_{k+1}, then $k \notin D(w)$. Thus we need to show that $\mathrm{maj}(\eta_{k+1}) = \mathrm{inv}(\gamma_{k+1}) = \mathrm{inv}(\gamma_k)$. In this case, the last letter in any compartment C of γ_k is the smallest letter in the compartment. So when we cyclically shift this compartment, we eliminate $|C| - 1$ inversions. But adding w_{k+1} to the end of γ_k will also introduce $|C| - 1$ inversions from that same compartment. Thus, there is no net change in the number of inversions between γ_k and γ_{k+1}, as desired.