Exercise 20. (a) Compute the signless Stirling numbers of the first kind $c(n,k)$ for $n = 1,2,3,4$ and $k = 1,\dots,n$ (i) directly, and (ii) using the recursion. Then give the Stirling numbers of the first kind $s(n,k)$ for $n = 1,2,3,4$ and $k = 1,\dots,n$.

Solution: Direct computations:

$$n = 1 : c(1,1) = |\{(1)\}| = 1.$$
$$n = 2 : c(2,1) = |\{(12)\}| = 1, c(2,2) = |\{(1)(2)\}| = 1.$$
$$n = 3 : c(3,1) = |\{(123),(132)\}| = 2, c(3,2) = |\{(12)(3),(13)(2),(23)(1)\}| = 3,$$
$$c(3,3) = |\{(1)(2)(3)\}| = 1.$$

For $n = 4$, let's be a little more clever.

$$c(4,1) = |\{w \in \mathcal{S}_4 \mid \text{type}(w) = (0,0,0,1)\}| = 3! = 6$$
$$\text{(every cycle starts with 4, followed by a permutation of } \{1,2,3\})$$
$$c(4,2) = |\{w \in \mathcal{S}_4 \mid \text{type}(w) = (1,0,1,0)\} \sqcup \{w \in \mathcal{S}_4 \mid \text{type}(w) = (0,2,0,0)\}|$$
$$= 4 * 2! + \binom{4}{2}/2 = 11$$
$$c(4,3) = |\{w \in \mathcal{S}_4 \mid \text{type}(w) = (2,1,0,0)\}| = \binom{4}{2} = 6$$
$$c(4,4) = |\{w \in \mathcal{S}_4 \mid \text{type}(w) = (4,0,0,0)\}| = 1$$

By recursion:

$$c(1,1) = c(0,0) + (1-1)c(0,1) = 1$$

$$c(2,1) = c(1,0) + (2-1)c(1,1) = 1$$
$$c(2,2) = c(1,1) + (2-1)c(1,2) = 1$$

$$c(3,1) = c(2,0) + (3-1)c(2,1) = 2 * 1 = 2$$
$$c(3,2) = c(2,1) + (3-1)c(2,2) = 1 + 2 * 1 = 3$$
$$c(3,3) = c(2,2) + (3-1)c(2,3) = 1$$

$$c(4,1) = c(3,0) + 3c(3,1) = 3 * 2 = 6$$
$$c(4,2) = c(3,1) + 4c(3,2) = 2 + 3 * 3 = 11$$
$$c(4,3) = c(3,2) + 3c(3,3) = 3 + 3 = 6$$
$$c(4,4) = c(3,3) + 3c(3,4) = 1$$

$n = 1$: $s(1,1) = 1$.
$n = 2$: $s(2,1) = -1$, $s(2,2) = 1$.
$n = 3$: $s(3,1) = 2$, $s(3,2) = -3$, $s(3,3) = 1$.
$n = 3$: $s(4,1) = -6$, $s(4,2) = 11$, $s(4,3) = -6$, $s(4,4) = 1$.

(b) Verify $\sum_{k=0}^{n} c(n,k)t^k = t(t+1)(t+2)\cdots(t+n-1)$ for $n = 0,1,2,3,4$.

Solution: $n=0$: $1 = c(0,0)\checkmark$ $n=1$: $t = c(1,1)t\checkmark$.
$n=2$: $t(t+1) = t^2 + t = c(2,2)t^2 + c(2,1)t\checkmark$.
$n=3$: $t(t+1)(t+2) = t^3 + 3t^2 + 2t = c(3,3)t^3 + c(3,2)t^2 + c(3,1)t\checkmark$
$n=3$: $t(t+1)(t+2)(t+3) = t^4 + 6t^3 + 11t^2 + 6t = c(4,4)t^4 + c(4,3)t^3 + c(4,2)t^2 + c(4,1)t\checkmark$.

Exercise 21. (Proving Proposition 1.3.7)

(a) Verify that $\sum_{k=0}^{n} c(n,k)t^k = n!Z_n(t,t,\ldots,t)$ for $n = 1,2,3$, and then explain why this identity holds in general.

Solution:

$$\begin{aligned}
1!Z_1(t) &= t\checkmark\\
2!Z_2(t,t) &= t_1^2 + t_2\big|_{t_i=t} = t^2 + t\checkmark\\
3!Z_3(t,t,t) &= t_1^3 + 3t_1t_2 + 2t_3\big|_{t_i=t} = t^3 + 3t^2 + 2t\checkmark\\
4!Z_4(t,t,t,t) &= t_1^4 + 6t_1^2t_2 + 3t_2^2 + 8t_3t_1 + 6t_4\big|_{t_i=t} = t^4 + 6t^3 + 11t^2 + 6t\checkmark
\end{aligned}$$

In general, the degree k monomials in $n!Z_n$ are in bijection with the permutations with k cycles, since the number of cycles in w is equal to the sum over i of $c_i(w)$. Thus, when we evaluate at $t_i = t$, the coefficient of the degree k term counts the number of permutations with k cycles.

(b) Cary out another example for the third proof of Proposition 1.3.7, again for $n = 9$ and $k = 4$.

Solution: Let $S = \{4,5,6,7,8\}$ and $f(4) = 4, f(5) = 3, f(6) = 1, f(7) = 7, f(8) = 1$. Then $T = \{6,7,8,9\}$ and $b_1 = 5$, $b_2 = 4$, $b_3 = 3$, $b_4 = 2$, $b_5 = 1$. We start to build w with four cycles, each starting with the elements of T:

$$(6\quad)(7\quad)(8\quad)(9\quad).$$

Then place b_i, one by one, so that there are $f(a_i) = b(n-b_i)$ numbers larger than b_i to the left of b_i (since we're adding them in decreasing order, there's no ambiguity about where to insert them):

$$\begin{aligned}
\text{insert 5 with 4 values to the left: } & (6\quad)(7\quad)(8\quad)(95\quad)\\
\text{insert 4 with 3 values to the left: } & (6\quad)(7\quad)(84\quad)(94\quad)\\
\text{insert 3 with 1 value to the left: } & (63\quad)(7\quad)(84\quad)(94\quad)\\
\text{insert 2 with 7 values to the left: } & (63\quad)(7\quad)(84\quad)(942\quad)\\
\text{insert 1 with 1 value to the left: } & (631)(7)(84)(942) = w
\end{aligned}$$

(c) Walk through and complete the third proof of Proposition 1.3.7.

Solution: Outline the proof given.

Completing the proof is a matter of (1) making sure the insertion algorithm is well-defined (never runs into trouble) and returns a permutation in $\mathcal{S}_{n,k}$, and (2) making sure the algorithm is invertible, and that that inverse always returns a subset $S \in \binom{[n-1]}{n-k}$ and function $f : S \to [n-1]$ with $f(i) \leq i$ whenever fed a permutation in $\mathcal{S}_{n,k}$.

For the first step, the algorithm produces k cycles by design; and when it runs, it places all n terms, so that it returns a permutation in $\mathcal{S}_{n,k}$. The place where we might run into trouble is if it is not possible to place b_i with $f(a_i)$ numbers to the left. But $a_i = n - b_i$, so that , and

$n - b_i$ is exactly the number of values (already placed) available which are larger than b_i. So the restriction on the function that $f(a_i) \leq a_i = n - b_i$ is exactly the right condition.

For the inverse, put $w \in \mathcal{S}_{n,k}$ in standard cycle notation. Then T is read off of the first values of each cycle of w; since n must be the largest value in its cycle, we always have $n \in T$. Then T determines S by

$$S = \{i \in [n-1] \mid n - i \notin T\}.$$

Since $|T| = k$, $|S| = n - |T| = n - k$, so $S \in \binom{[n-1]}{n-k}$. Finally, $([n] - T)_> = \{b_1, \ldots, b_{n-k}\}$ is the set of numbers which do not start the cycles, placed in decreasing order. So set $f(a_i) = f_(n - b_i) = \#\{\text{values larger than } b_i \text{ to the left of } b_i \text{ in } w\}$. Since there are at most $n - b_i$ such values, and at least one such value exists (b_i doesn't start a cycle, by definition), we have $f : S \to [n-1]$ and $f(a_i) \leq a_i$, as desired.

(d) Read the fourth proof and example 1.3.9. Cary out another example for $n = 9$ and $t = 4$ for a sequence $(a_1, \ldots, a_n)$ of your choice.

Solution: If $n = 9$, $t = 4$, then $t+n-i-1 = 12-i$ and $t-1 = 3$. Let $a = (0, 10, 5, 2, 0, 1, 1, 4, 2)$. Then the insertion algorithm goes as follows:

$$\begin{array}{ll}
(9) & f(C_1) = 2 + 1 = 3 \\
a_8 = 4 = t + 0(98) & \\
a_7 = 1 \leq 3(7)(98) & f(C_2) = 1 + 1 = 2 \\
a_6 = 1 \leq 3(6)(7)(98) & f(C_3) = 2 \\
a_5 = 0 \leq 3(5)(6)(7)(98) & f(C_4) = 1 \\
a_4 = 2 \leq 3(4)(5)(6)(7)(98) & f(C_5) = 3 \\
a_3 = 5 = t + 1(4)(53)(6)(7)(98) & \\
a_2 = 10 = t + 6(4)(52)(6)(7)(982) & \\
a_1 = 0 \leq 0(1)(4)(52)(6)(7)(982) & f(C_6) = 1
\end{array}$$

Exercise 22. Using only the combinatorial definitions of the signless Stirling numbers $c(n,k)$, give formulas for $c(n,1)$, $c(n,n)$, $c(n,n-1)$, and $c(n,n-2)$.

Solution: The permutations of $[n]$ with 1 cycle are in bijection with the permutations of $n-1$ (start the cycle with n, and finish it with a permutation of $[n-1]$), so $c(n,1) = (n-1)!$.

The only permutation of $[n]$ with n cycles is the identity permutation, so $c(n,n) = 1$.

If a permutation of $[n]$ has $n-1$ cycles, that means that it must be of type $(n-2, 1, 0, \ldots, 0)$. So $c(n, n-1) = \binom{n}{2}$ (choose the two elements to go into the 2-cycle).

If a permutation of $[n]$ has $n-2$ cycles, then it's either of type $(n-3, 0, 1, 0, \ldots, 0)$, of which there are $\binom{n}{3} * 2$ (choose the three elements to go into the 3-cycle, and then there are 2 permutations that have those three elements in that 3-cycle), or it's of type $(n-4, 2, 0, \ldots, 0)$, of which there are $\binom{n}{4} * 3$ (choose the 4 elements to go into the two 2-cycles, and then there are 3 ways to distribute those 4 elements into two 2-cycles). So $c(n, n-2) = 2\binom{n}{3} + 3\binom{n}{4}$.

Exercise 23. Inversions and descents.

(a) For each of $w \in \mathcal{S}_3$, write w in word form and give (i) w^{-1}, (ii) $I(w)$, (iii) $\mathrm{inv}(w)$, (iv) $\mathrm{code}(w)$, (v) $D(w)$, (vi) $\mathrm{des}(w)$, and (vii) $\mathrm{maj}(w)$. (Make a table.)

Solution:

$\mathcal{S}_3$	123	132	213	231	312	321
w^{-1}	123	132	213	312	231	321
$I(w)$	$(0,0,0)$	$(0,1,0)$	$(1,0,0)$	$(2,0,0)$	$(1,1,0)$	$(2,1,0)$
$\mathrm{code}(w)$	$(0,0,0)$	$(0,1,0)$	$(1,0,0)$	$(1,1,0)$	$(2,0,0)$	$(2,1,0)$
$\mathrm{inv}(w)$	0	1	1	2	2	3
$D(w)$	$\emptyset$	$\{2\}$	$\{1\}$	$\{2\}$	$\{1\}$	$\{1,2\}$
$\mathrm{des}(w)$	0	1	1	1	1	2
$\mathrm{maj}(w)$	0	2	1	2	1	3

(b) Use your calculations in (a) to verify

(i) $I(w^{-1})$ and $\mathrm{code}_i(w) = \#\{j > i \mid w(j) < w(i)\}$ are equivalent definitions of $\mathrm{code}(w)$, CHECK

(ii) Corollary 1.3.13,

Solution:

$$\sum_{w \in \mathcal{S}_3} = q^0 + q + q + q^2 + q^2 + q^3 = 1 + 2q + 2q^2 + q^3 = (1+q)(1+q+q^3)\checkmark$$

(iii) Proposition 1.3.14,

Solution: The only permutation to check here is that $\mathrm{inv}(231) = 2 = \mathrm{inv}(312)$ (the rest of the permutations are equal to their own inverses).

(iv) the proof of Prop 1.3.14 (show the bijection between inversions (i,j) in w and inversions $(w_i^{-1}, w{-}1_j)$ in w^{-1});

Solution: For 132, 213, and 321, the inversion labels are exactly the same as the inversion places (and these permutations are self-inverses). For 231, the inversions are $(2,1) = (w_1, w_3)$ and $(3,1) = (w_2, w_3)$, where the inversions in 312 are $(3,1) = (w_1, w_2)$ and $(3,2) = (w_1, w_3)$.

(v) equation (1.41),

Solution: There is 1 permutation with $\mathrm{inv}(w) = 0$, 2 permutations with $\mathrm{inv}(w) = 1$, 2 permutations with $\mathrm{inv}(w) = 2$, and 1 with $\mathrm{inv}(w) = 3$. Similarly, there is 1 permutation with $\mathrm{maj}(w) = 0$, 2 permutations with $\mathrm{maj}(w) = 1$, 2 permutations with $\mathrm{maj}(w) = 2$, and 1 with $\mathrm{maj}(w) = 3$.

for $n = 3$.