Exercise 16. For each of the following permutations of [9], give whichever of the following is not already given.

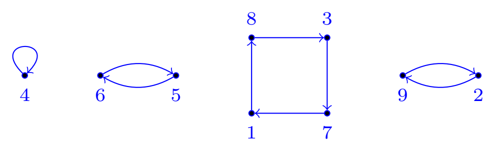
(i) The function representation.
(ii) The word representation.
(iii) The standard cycle representation.
(iv) The digraph representation.
(v) The word given by the fundamental bijection (the ˆ word).
(vi) The diagrammatic representation.

(a) $w : [9] \to [9]$, the permutation given by

$$1 \mapsto 8, \quad 2 \mapsto 9, \quad 3 \mapsto 7, \quad 4 \mapsto 4, \quad 5 \mapsto 6, \quad 6 \mapsto 5, \quad 7 \mapsto 1, \quad 8 \mapsto 3, \quad 9 \mapsto 2.$$

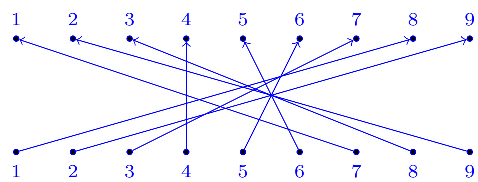
Solution:

function: (given)
word: 897465132
st. cycle: (4)(65)(8371)(92)
digraph:



$\hat{w}$: 465837192

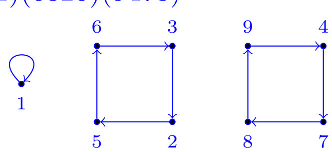
diagram:



(b) $v = (6325)(1)(9478)$.

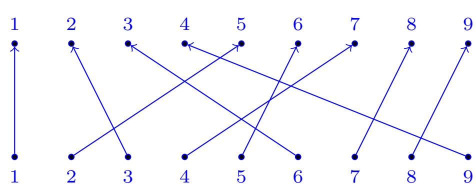
Solution:

function: $1 \mapsto 1, 2 \mapsto 5, 3 \mapsto 2, 4 \mapsto 7, 5 \mapsto 6, 6 \mapsto 3, 7 \mapsto 8, 8 \mapsto 9, 9 \mapsto 4$
word: 25763894
st. cycle: (1)(6325)(9478)
digraph:



$\hat{v}$: 163259478

diagram:



(c) The permutation u determined by $\hat{u} = 123456798$.

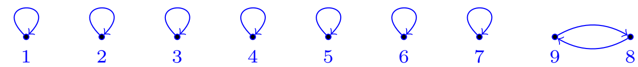
Solution:

function: $i \mapsto i$ for $i = 1, \ldots, 7, 8 \mapsto 9, 9 \mapsto 8$

word: 123456798

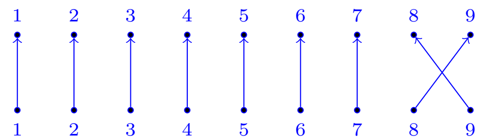
st. cycle: (1)(2)(3)(4)(5)(6)(7)(98)

digraph:



$\hat{u}$: (given)

diagram:



Exercise 17. For each of the permutations in 16, give the cycle type, and the number of permutations of [9] that have the same cycle type.

Solution: The permutation w has cycle type $(1,2,0,1,0,0,0,0,0)$; there are $9!/(2^2 * 2! * 4)$ permutations of 9 with this cycle type.
The permutation v has cycle type $(1,0,0,2,0,0,0,0,0)$; there are $9!/(4^2 * 2!)$ permutations of 9 with this cycle type.
The permutation u has cycle type $(7,1,0,0,0,0,0,0,0)$; there are $9!/(7!)$ permutations of 9 with this cycle type.

Additionally, give the following.

(a) For w in 16(a), verify the equation $n = \sum_i ic_i$.

Solution:

$$\sum_i ic_i = 1 * 1 + 2 * 2 + 4 * 1 = 1 + 4 + 4 = 9 \checkmark$$

(b) For v in 16(b), verify that v has the same number of cycles as $\hat{v}$ has left-to-right maxima (be sure to identify the left-to-right maxima).

Solution: The left-to-right maxima of $\hat{v}$ are $1, 6$, and 9, of which there are three, the same number of cycles that v has. ✓

(c) For u in 16(c), what is $t^{\mathrm{type}(u)}$?

Solution: Since $\mathrm{type}(u) = (7,1,0,0,0,0,0,0,0)$, we have $t^{\mathrm{type}(u)} = t_1^7 t_2$.

Exercise 18. Show that the number of permutations $w \in \mathcal{S}_n$ fixed by the fundamental bijection $\mathcal{S}_n \xrightarrow{\hat{}} \mathcal{S}_n$ (i.e. $|\{w \in \mathcal{S}_n \mid \hat{w} = w\}|$) is the Fibonacci number f_{n+1}.

Solution: Let

$$a_n = |\{w \in \mathcal{S}_n \mid \hat{w} = w\}|.$$

Write $w \in \mathcal{S}_n$ in standard cycle form,

$$w = (a_1 a_2 \cdots a_{\ell_1})(a_{\ell_1+1} \cdots a_{\ell_2}) \cdots (a_{\ell_{k-1}+1} \cdots a_n).$$

This means that w maps

$$a_{\ell_{k-1}+1} \mapsto a_{\ell_{k-1}+2}, \quad a_{\ell_{k-1}+2} \mapsto a_{\ell_{k-1}+3}, \quad \ldots, \quad a_n \mapsto a_{\ell_{k-1}+1},$$

and that $a_{\ell_{k-1}+1} > a_i$ for $i > \ell_{k-1}+1$. Further,

$$\hat{w} = a_1 a_2 \cdots a_{\ell_1} a_{\ell_1+1} \cdots a_{\ell_2} \cdots a_{\ell_{k-1}+1} \cdots a_n$$

in word form, so that $\hat{w}$ maps

$$\ell_{k-1}+1 \mapsto a_{\ell_{k-1}+1}, \quad \ell_{k-1}+2 \mapsto a_{\ell_{k-1}+2}, \quad \ldots, \quad n \mapsto a_n.$$

So if $w = \hat{w}$, we have

$$a_n = \ell_{k-1}+1, \quad a_{\ell_{k-1}+1} = \ell_{k-1}+2, \quad \ldots, \quad a_{n-1} = n.$$

But since $a_{\ell_{k-1}+1} > a_i$ for $i > \ell_{k-1}+1$, this is only a contradiction if $\ell_{k-1}+1 = n$ or $n-1$, so that the last cycle is either (n) or $(n, n-1)$. Recursively applying the same reasoning, we have that $w = \hat{w}$ if any only if $w = C_1 \cdots C_k$, where $C_k = (n)$ or $(n, n-1)$ and $w' = C_1 \cdots C_{k-1}$ satisfies $\widehat{w'} = w'$. So the good permutations of n ending in (n) are in bijection with good permutations of $n-1$, and the good permutations of n ending in $(n, n-1)$ are in bijection with good permutations of $n-2$. Thus

$$a_n = a_{n-1} + a_{n-2}.$$

Finally, $a_1 = |\{(1)\}| = 1 = f_2$ and $a_2 = |\{(1)(2), (21)\}| = 2 = f_3$, so $a_n = f_{n+1}$.

Exercise 19.

(a) For $Z_n = \frac{1}{n!}\sum_{w \in \mathcal{S}_n} t^{\text{type}(w)}$, calculate Z_1, Z_2, Z_3, and Z_4 explicitly (verifying the formulas between (1.25) and (1.26) in EC1).

Solution:

$\mathcal{S}_n$	Z_n
$\{(1)\}$	$1(t_1^1) = t$
$\{(1)(2), (12)\}$	$\frac{1}{2!}\left(t_1^2 + t_2\right)$
$\{(1)(2)(3), (21)(3), (2)(31), (1)(32), (312), (321)\}$	$\frac{1}{3!}\left(t_1^3 + 3t_1t_2 + 2t_3\right)$
$\{(1)(2)(3)(4), (21)(3)(4), (2)(31)(4),$ $(1)(32)(4), (2)(3)(41), (1)(3)(42), (1)(2)(43),$ $(21)(43), (31)(42), (32)(41), (312)(4), (321)(4), (412)(3),$ $(421)(3), (413)(2), (431)(2), (423)(1), (432)(1)$ $(4123), (4132), (4213), (4231), (4312), (4321)\}$	$\frac{1}{4!}\left(t_1^4 + 6t_1^2t_2 + 3t_2^2 + 8t_1t_3 + 6t_4\right)$

(b) For $E_k(n) = \frac{1}{n!}\sum_{w \in \mathcal{S}_n} c_k(w)$, verify that

$$E_k(n) = \frac{\partial}{\partial t_k} Z_n(t_1, t_2, \ldots, t_n)\Big|_{\substack{t_i=1 \\ i=1,\ldots,n}}.$$

Solution:

$$\frac{\partial}{\partial t_k} Z_n(t_1, t_2, \ldots, t_n)\Big|_{\substack{t_i=1\\ i=1,\ldots,n}} = \frac{1}{n!}\sum_{w\in\mathcal{S}_n} \frac{\partial}{\partial t_k} t_1^{c_1(w)} \cdots t_k^{c_k(w)} \cdots t_n^{c_n(w)}\Big|_{\substack{t_i=1\\ i=1,\ldots,n}}$$

$$= \frac{1}{n!}\sum_{w\in\mathcal{S}_n} c_k(w) t_1^{c_1(w)} \cdots t_k^{c_k(w)-1} \cdots t_n^{c_n(w)}\Big|_{\substack{t_i=1\\ i=1,\ldots,n}}$$

$$= \frac{1}{n!}\sum_{w\in\mathcal{S}_n} c_k(w) = E_k(n).$$

(c) Give a combinatorial proof of $E_k(n) = 1/k$ by (i) explaining why there are $\binom{n}{k}(k-1)!$ k-cycles, (ii) explaining why each k-cycle appears in $(n-k)!$ permutations, and (iii) computing $E_k(n)$ using these two values.

Solution: The build a k-cycle in a permutation of n, you first pick the k elements, of which there are $\binom{n}{k}$ choices. Then order those elements; since all cyclic rotations are the same, fix the largest element first, and then there are $(k-1)!$ orderings of the remaining elements. So there are $\binom{n}{k}(k-1)!$ possible k-cycles.

If a given k-cycle appears in a permutation, that fixes k values of $w(i)$. Therefore there are $(n-k)!$ choices for the remaining values of $w(i)$.

Thus

$$\sum_{w\in\mathcal{S}_n} c_k(w) = \binom{n}{k}(k-1)!(n-k)! = \frac{n!}{k!(n-k)!}(k-1)!(n-k)! = \frac{n!}{k},$$

and so

$$E_k(n) = \frac{1}{n!}\sum_{w\in\mathcal{S}_n} c_k(w) = \frac{1}{n!}\frac{n!}{k} = \frac{1}{k}.$$