# Solutions for HW4

**Exercise 13.**

(a) Prove

$$(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots(1+x_n+x_n^2+\cdots) = \sum_{M=(S,\nu)} \prod_{x_i\in S} x_i^{\nu(x_i)},$$

by induction on $n$.

*Solution:* First note that since the multisets on $\{x_1\}$ are determined by $\nu(x_1)$, the set of multisets on $\{x_1\}$ is in bijection with $\mathbb{N}$. So

$$1+x_1+x_1^2+\cdots = \sum_{k\in\mathbb{N}} x_1^k = \sum_{M=(\{x_1\},\nu)} x_1^{\nu(x_1)} = \sum_{M=(\{x_1\},\nu)} \prod_{x_i\in\{x_1\}} x_i^{\nu(x_i)},$$

so our identity holds for $n=1$.

Now fix $n$ and assume, for $S=\{x_1,\ldots,x_n\}$, we have

$$(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots(1+x_n+x_n^2+\cdots) = \sum_{M=(S,\nu)} \prod_{x_i\in S} x_i^{\nu(x_i)}.$$

Then with $S' = S \sqcup \{x_{n+1}\}$,

$$\begin{aligned}
&(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots(1+x_{n+1}+x_{n+1}^2+\cdots)\\
&\overset{\text{IHOP}}{=} (1+x_{n+1}+x_{n+1}^2+\cdots) \sum_{M=(S,\nu)} \prod_{x_i\in S} x_i^{\nu(x_i)}\\
&= \sum_{k\in\mathbb{N}} \sum_{M=(S,\nu)} \left( \prod_{x_i\in S} x_i^{\nu(x_i)} \right) x_{n+1}^k\\
&= \sum_{M=(S',\nu)} \prod_{x_i\in S'} x_i^{\nu(x_i)}.
\end{aligned}$$

So our identity holds for all $n$ by induction.

(b) Show algebraically that $\binom{-n}{k}(-1)^k = \binom{n+k-1}{k}$.

*Solution:*

$$\binom{-n}{k}(-1)^k = \frac{1}{k!}(-1)^n(-n)(-n-1)\ldots(-n-k+1) = \frac{1}{k!}n(n+1)\cdots(n+k-1) = \binom{n+k-1}{k}.$$

(c) (a) Write the generating function (both series and closed form) for the number of weak compositions of $n$ with $k$ parts.
[Hint: This should look something like the generating function for multisets.]

*Solution:* Since the number of weak compositions of $n$ with $k$ parts is

$$\binom{n+k-1}{k-1} = \left(\!\binom{k-1}{n}\!\right),$$

the generating function for weak compositions of $n$ with $k$ parts is

$$\sum_{n\in\mathbb{N}} \binom{n+k-1}{k-1} x^n = (1-x)^{-k}.$$

(b) Write the generating function (both series and closed form) for the number of (not weak) compositions of $n$ with $k$ parts.

*Solution:* Since the number of compositions of $n$ with $k$ parts is $\binom{n-1}{k-1}$, the generating function for compositions of $n$ with $k$ parts is

$$\sum_{k=1}^{n} \binom{n-1}{k-1} x^k = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^{\ell+1} = x(1+x)^{n-1}.$$

(c) Write the generating function for the number of weak compositions of $n$ with $k$ parts, all less than $j$.

*Solution:* Let $\kappa(n,j,k)$ be the number of weak compositions of $n$ with $k$ parts, all less than $j$. Start with

$$(1+x_1+x_1^2+\cdots+x_1^{j-1})(1+x_2+x_2^2+\cdots+x_2^{j-1})\cdots(1+x_k+x_k^2+\cdots+x_n^{j-1})$$
$$= \sum_{n\in\mathbb{N}} \sum_{\substack{\alpha_1+\cdots+\alpha_k=n\\ 1\le\alpha_i\le j-1}} x_1^{\alpha_1}\cdots x_k^{\alpha_k}.$$

Then evaluating at $x_1=\cdots=x_k=x$, we get

$$\begin{aligned}\sum_{n\in\mathbb{N}} \kappa(n,j,k)x^n &= \sum_{n\in\mathbb{N}} \sum_{\substack{\alpha_1+\cdots+\alpha_k=n\\ 1\le\alpha_i\le j-1}} x^n \\ &= (1+x+x^2+\cdots+x^{j-1})^k \\ &= (1-x^j)^k(1-x)^{-k}.\end{aligned}$$

(d) Item give a generating function proof that the number of weak compositions of $n$ into $k$ parts, with each part less than $j$, is

$$\sum_{\substack{r,s\in\mathbb{N}\\ r+sj=n}} (-1)^s \binom{k+r-1}{r}\binom{k}{s}.$$

*Solution:* Continuing from the previous part, use the generalized binomial theorem to expand

$$\begin{aligned}\sum_{n\in\mathbb{N}} \kappa(n,j,k)x^n &= (1-x^j)^k(1-x)^{-k} \\ &= \left(\sum_{s\in\mathbb{N}} \binom{k}{s}(-x^j)^s\right)\left(\sum_{r\in\mathbb{N}} \binom{-k}{r}(-x)^r\right) \\ &= \left(\sum_{s\in\mathbb{N}} \binom{k}{s}(-1)^s x^{sj}\right)\left(\sum_{r\in\mathbb{N}} \binom{-k}{r}(-1)^r x\right) \\ &= \left(\sum_{s\in\mathbb{N}} \binom{k}{s}(-1)^s x^{sj}\right)\left(\sum_{r\in\mathbb{N}} \binom{k+r-1}{r} x\right),\end{aligned}$$

since $\binom{-k}{r}(-1)^r = \binom{k+r-1}{r}$. Now using the multiplication rule for series, we have that the coefficient of $x^n$ on the last line is

$$\sum_{\substack{r,s\in\mathbb{N}\\ r+sj=n}} \binom{k}{s}(-1)^s\binom{k+r-1}{r},$$

thus proving the desired result.

**Exercise 14.**

(a) **Deriving multinomial coefficients algebraically.** Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a composition of $n$. Use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to compute $\binom{n}{\alpha_1,\ldots,\alpha_\ell}$, noting that you can first choose the $\alpha_1$ items from $n$, then $\alpha_2$ from $n-\alpha_1$, then $\alpha_3$ from $n-(\alpha_1+\alpha_2)$, and so on.

*Solution:* By first choosing the $\alpha_1$ items from $n$, then $\alpha_2$ from $n-\alpha_1$, then $\alpha_3$ from $n-(\alpha_1+\alpha_2)$, and so on, product rule says that

$$\begin{aligned}
&\binom{n}{\alpha_1,\ldots,\alpha_\ell}\\
&\quad= \binom{n}{\alpha_1}\binom{n-\alpha_1}{\alpha_2}\binom{n-(\alpha_1+\alpha_2)}{\alpha_3}\cdots\binom{n-(\alpha_1+\cdots\alpha_{\ell-1})}{\alpha_\ell}\\
&\quad= \frac{n!}{\alpha_1!(n-\alpha_1)!}\frac{(n-\alpha_1)!}{\alpha_2!(n-(\alpha_1+\alpha_2))!}\frac{(n-(\alpha_1+\alpha_2))!}{\alpha_3!(n-(\alpha_1+\alpha_2+\alpha_3))!}\cdots\frac{(n-(\alpha_1+\cdots+\alpha_{\ell-1}))!}{\alpha_\ell!(n-(\alpha_1+\cdots+\alpha_\ell))!}\\
&\quad= \frac{n!}{\alpha_1!\cdots\alpha_\ell!0!} = \boxed{\frac{n!}{\alpha_1!\cdots\alpha_\ell!}},
\end{aligned}$$

since $n-(\alpha_1+\cdots+\alpha_\ell) = n-n = 0$.

(b) **Multinomial theorem.** Following our proof of the binomial theorem, show that

$$(x_1+x_2+\cdots+x_\ell)^n = \sum_{\substack{(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{N}^\ell\\ \alpha_1+\cdots+\alpha_\ell=n}} \binom{n}{\alpha_1,\ldots,\alpha_\ell} x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_\ell^{\alpha_\ell}.$$

[Hint: Recall that the key computation for the binomial theorem was that, for $S = \{x^{(1)},\ldots,x^{(n)}\}$,

$$\prod_{x^{(i)}\in S}(1+x^{(i)}) = \sum_{T\subseteq S}\prod_{x^{(i)}\in T} x^{(i)}, \qquad \text{so that} \qquad (1+x)^n = \sum_{T\subseteq S}\prod_{x^{(i)}\in T} x = \sum_{T\subseteq S} x^{|T|}.$$

The former we had to prove by induction on $n$. Now fix $\ell$, and let $S = \{x_i^{(j)} \mid 1\le i\le \ell, 1\le j\le n\}$ (so that there are $n$ distinct variables associated to each $x_i$), and walk through a similar proof.]

*Solution:* First consider the case where $n=1$, so that

$$\sum_{\substack{(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{N}^\ell\\ \alpha_1+\cdots+\alpha_\ell=1}} \binom{1}{\alpha_1,\ldots,\alpha_\ell} x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_\ell^{\alpha_\ell} = \sum_{i=1}^{\ell} x_1^0\cdots x_{i-1}^0 x_i^1 x_{i+1}^0\cdots x_\ell^0 = \sum_{i=1}^{\ell} x_i. \checkmark$$

Now assume

$$(x_1+x_2+\cdots+x_\ell)^{n-1} = \sum_{\substack{(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{N}^\ell\\ \alpha_1+\cdots+\alpha_\ell=n-1}} \binom{n-1}{\alpha_1,\ldots,\alpha_\ell} x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_\ell^{\alpha_\ell}$$

for a fixed $n$. Note that since $\binom{n}{\alpha_1,\ldots,\alpha_\ell}$ gives the number of ways of partitioning $[n]$ into labeled $\ell$ sets of size $\alpha_1, \ldots, \alpha_\ell$, respectively, by tracking which set $n$ goes into, we have that

$$\binom{n}{\alpha_1,\ldots,\alpha_\ell} = \sum_{i=1}^{\ell} \binom{n-1}{\alpha_1,\ldots,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},\cdots,\alpha_\ell} \tag{$*$}$$

(this is the multinomial coefficient generalization of the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n}{k}$, since $\binom{n}{k} = \binom{n}{k,n-k}$). So

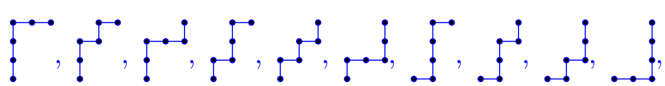
$$\begin{aligned}
(x_1+x_2+\cdots+x_\ell)^n &\overset{\text{IHOP}}{=} \left(\sum_{i=1}^{\ell} x^i\right) \sum_{\substack{(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{N}^\ell \\ \alpha_1+\cdots+\alpha_\ell=n-1}} \binom{n-1}{\alpha_1,\ldots,\alpha_\ell} x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_\ell^{\alpha_\ell} \\
&= \sum_{i=1}^{\ell} \sum_{\substack{(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{N}^\ell \\ \alpha_1+\cdots+\alpha_\ell=n-1}} \binom{n-1}{\alpha_1,\ldots,\alpha_\ell} x_i(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_\ell^{\alpha_\ell}) \\
&= \sum_{\substack{(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{N}^\ell \\ \alpha_1+\cdots+\alpha_\ell=n-1}} \sum_{i=1}^{\ell} \binom{n-1}{\alpha_1,\ldots,\alpha_\ell} x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}x_i^{\alpha_i+1}x_{i+1}^{\alpha_{i+1}}\cdots x_\ell^{\alpha_\ell} \\
&= \sum_{\substack{(\alpha'_1,\ldots,\alpha'_\ell)\in\mathbb{N}^\ell \\ \alpha'_1+\cdots+\alpha'_\ell=n}} \sum_{i=1}^{\ell} \binom{n-1}{\alpha_1,\ldots,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},\cdots,\alpha_\ell} x_1^{\alpha'_1}x_2^{\alpha'_2}\cdots x_\ell^{\alpha'_\ell} \\
&= \sum_{\substack{(\alpha'_1,\ldots,\alpha'_\ell)\in\mathbb{N}^\ell \\ \alpha'_1+\cdots+\alpha'_\ell=n}} \binom{n+1}{\alpha'_1,\ldots,\alpha'_\ell} x_1^{\alpha'_1}x_2^{\alpha'_2}\cdots x_\ell^{\alpha'_\ell},
\end{aligned}$$

by $(*)$.

(c) **Lattice paths.** Proposition 1.2.1 in EC1 says the following. Let $v = (a_1,\ldots,a_d) \in \mathbb{N}^d$, and let $e_i$ denote the $i$th unit coordinate vector in $\mathbb{Z}^d$. The number of lattice paths in $\mathbb{Z}^d$ from the origin $(0,0,\ldots,0)$ to $v$ with steps in $\{e_1,\ldots,e_d\}$ is given by the multinomial coefficient $\binom{a_1+\cdots+a_d}{a_1,\ldots,a_d}$.

(i) Check this proposition for $d = 2$ with the point $v = (2,3)$.

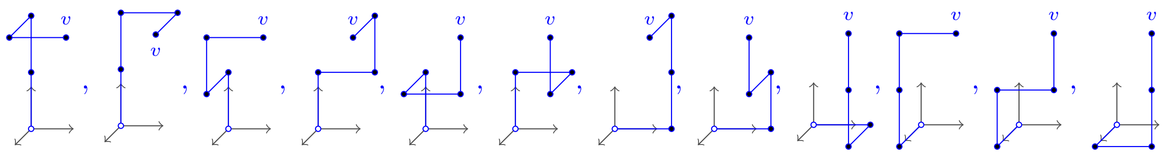
*Solution:* The lattice paths from $(0,0)$ to $(2,3)$ with steps in $S = \{(1,0),(0,1)\}$ are



of which there are $10 = \frac{(2+3)!}{2!3!} = \binom{2+3}{2,3}$. ✓

(ii) Check this proposition for $d = 3$ with the point $v = (1,1,2)$.

*Solution:* The lattice paths from $(0,0,0)$ to $(1,1,2)$ with steps in $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ are




of which there are $12 = \frac{(1+1+2)!}{1!1!2!} = \binom{1+1+2}{1,1,2}$. ✓

(iii) Prove this theorem (spell out the book's proof with more details).

*Solution:* Let $v_0, \ldots, v_k$ be a lattice path from 0 to $v_k = (a_1, \cdots, a_d)$ with elementary steps, where $k = a_1 + \cdots + a_d$. Then consider the sequence determined by the path

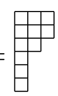
$$(v_1 - v_0, v_2 - v_1, \ldots, v_k - v_{k-1}) = (e_{i_1}, e_{i_2}, \ldots, e_{i_k}) \in \mathcal{S}_M,$$

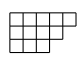
where $M$ is the multiset on $S = \{e_1, \ldots, e_d\}$ with weight $\nu(e_i) = a_i$. Similarly, each sequence $(e_{i_1}, e_{i_2}, \ldots, e_{i_k}) \in \mathcal{S}_M$ determines the path

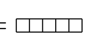
$$0, e_{i_1}, e_{i_1} + e_{i_2}, \ldots, e_{i_1} + \cdots + e_{i_k},$$

from 0 to $e_{i_1} + \cdots + e_{i_k} = (a_1, \ldots, a_d)$. So the lattice paths from 0 to $v_k = (a_1, \cdots, a_d)$ with elementary steps are in bijection with permutations in $\mathcal{S}_M$, of which there are $\binom{a_1+\cdots+a_d}{a_1,\ldots,a_d}$.

(d) **Integer partitions.** An (integer) partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of $n$ is a composition of $n$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. We draw partitions as $n$ boxes piled up and to the left into a corner, with $\lambda_1$ boxes in the first row, $\lambda_2$ boxes in the second row, and so on. For example,

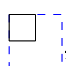
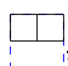
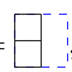
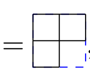
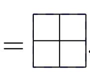
$\lambda = (3, 3, 2, 1, 1, 1) =$  is a partition of 11,

$\lambda = (5, 4, 3) =$  is a partition of 12,

$\lambda = (5) =$  is a partition of 5, and

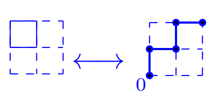
$\lambda = \emptyset$ is a partition of 0.

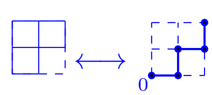
The six partitions to fit in a $2 \times 2$ square are

$\emptyset =$ , $(1) =$ , $(2) =$ , $(1,1) =$ , $(2,1) =$ , and $(2,2) =$ .

Use lattice paths to count the number of integer partitions fitting into a $m \times n$ rectangle.

*Solution:* Note that by taking a partition inside an $m \times n$ rectangle, and overlaying an $m \times n$ grid with the origin $(0,0)$ at the south-west corner, we have that tracing the south-east wall of the partition, tracing the wall of the grid when appropriate, determines a lattice path from $(0,0)$ to $(n,m)$ with elementary steps. Vice versa, any lattice path from $(0,0)$ to $(n,m)$ with elementary steps determines the partition whose $i$th part ends at the $m - i + 1$st up-step. For example,



and



Thus, partitions inside an $m \times n$ rectangle are in bijection with lattice paths from $(0,0)$ to $(n,m)$ with elementary steps, of which there are $\boxed{\binom{n+m}{n,m}}$.

**Exercise 15.**

(a) For each of the following, give examples for small values of $n$. Then express the following numbers in terms of the Fibonacci numbers.

(i) **Example:** The number of subsets $S$ of the set $[n] = \{1, 2, \ldots, n\}$ such that $S$ contains no two consecutive integers.

**Answer:** Let $a_n$ be the number of good subsets of $[n]$. Note that $a_1 = |\{\emptyset, \{1\}\}| = 2$ and $a_2 = \{\emptyset, \{1\}, \{2\}\} = 3$.
Now divide $S$ into 2 cases: either it contains $n$ or it doesn't. Since every good subset without $n$ is also a good subset of $[n-1]$, and vice versa, the number of good subsets without $n$ is $a_{n-1}$. Similarly $S \mapsto S - \{n\}$ is a bijection between good subsets of $[n]$ containing $n$ and good subsets of $[n-2]$, the number of good subsets of $[n]$ containing $n$ is $a_{n-2}$. So $a_n = a_{n-1} + a_{n-2}$, with $a_1 = 2 = f_3$, $a_2 = 3 = f_4$. This is the same recurrence that determines $f_n$, but shifted so that $a_n = f_{n+2}$.
So there are $\boxed{f_{n+2}}$ good subsets of $[n]$. □

NOTE: For many of these, this is the strategy you want. Make a recurrence relation that looks like the Fibonacci recurrence, and shift appropriately. For at least one, you'll want to use a previous part.

(ii) The number of compositions of n into parts greater than 1.

*Solution:* Let

$S_n = \{$ compositions of $n$ into parts greater than 1 $\}$,

$A_n = \{$ compositions of $n$ into parts greater than 1, whose last part is 2 $\}$, and

$B_n = \{$ compositions of $n$ into parts greater than 1, whose last part is note 2 $\}$,

so that $a_n = |S_n|$ is the value we wish to enumerate, and

$$S_n = A_n \sqcup B_n, \qquad \text{so that } a_n = |A_n| + |B_n|.$$

Note that $a_0 = a_1 = 0$, and $a_2 = |\{(2)\}| = 1$.
The function from $A_n$ which takes $(\alpha_1, \ldots, \alpha_\ell, 2)$ to $(\alpha_1, \ldots, \alpha_\ell)$ is a function from $A_n$ to $S_{n-2}$ since $\alpha_1 + \cdots + \alpha_\ell + 2 = n$ and $\alpha_i > 1$. Its inverse, appending a good composition of $n-2$ by 2, is well-defined, so $A_n$ is in bijection with $S_{n-2}$.
Similarly, the function from $B_n$ which takes $(\alpha_1, \ldots, \alpha_\ell)$ to $(\alpha_1, \ldots, \alpha_\ell - 1)$ is a function from $B_n$ to $S_{n-1}$, since $(\alpha_1 + \cdots + \alpha_\ell) - 1 = n - 1$ and $\alpha_i > 1$ for $i = 1, \ldots, \ell = 1$, and $\alpha_\ell > 2$ so $\alpha_\ell - 1 > 1$. Its inverse, adding 1 to the last part, is well-defined, so $B_n$ is in bijection with $S_{n-1}$.
Thus

$$a_n = |S_n| = |A_n| + |B_n| = |S_{n-2}| + |S_{n-1}| = a_{n-2} + a_{n-1}.$$

Since $a_1 = 0 = f_0$ and $a_2 = 1 = f_1$, we have $\boxed{a_n = f_{n+1}}$.

(iii) The number of compositions of n into parts equal to 1 or 2.

*Solution:* Let

$S_n = \{$ compositions of $n$ into into parts equal to 1 or 2 $\}$,

$A_n = \{$ compositions in $S_n$, whose last part is 1 $\}$, and

$B_n = \{$ compositions in $S_n$, whose last part is 2 $\}$, so that $a_n = |S_n|$ is the value we wish to enumerate, and

$$S_n = A_n \sqcup B_n, \qquad \text{so that } a_n = |A_n| + |B_n|.$$

Note that $a_0 = 0$, and $a_1 = |\{(1)\}| = 1$.
The function from $A_n$ which takes $(\alpha_1, \dots, \alpha_\ell, 1)$ to $(\alpha_1, \dots, \alpha_\ell)$ is a function from $A_n$ to $S_{n-1}$ since $(\alpha_1 + \cdots + \alpha_\ell) + 1 = n$ and $\alpha_i = 1$ or 2. Its inverse, appending a good composition of $n-1$ by 1, is well-defined, so $A_n$ is in bijection with $S_{n-1}$.
Similarly, the function from $B_n$ which takes $(\alpha_1, \dots, \alpha_\ell, 2)$ to $(\alpha_1, \dots, \alpha_\ell)$ is a function from $B_n$ to $S_{n-2}$, since $(\alpha_1 + \cdots + \alpha_\ell) + 2 = n$ and $\alpha_i = 1$ or 2. Its inverse, appending a good composition of $n-2$ by 2, is well-defined, so $B_n$ is in bijection with $S_{n-2}$.
Thus

$$a_n = |S_n| = |A_n| + |B_n| = |S_{n-1}| + |S_{n-2}| = a_{n-1} + a_{n-2}.$$

Since $a_0 = 0 = f_0$ and $a_1 = 1 = f_1$, we have $\boxed{a_n = f_n}$.

(iv) The number of compositions of n into odd parts.

*Solution:* Let

$$S_n = \{ \text{ compositions of } n \text{ into into odd parts } \},$$
$$A_n = \{ \text{ compositions in } S_n, \text{ whose last part is 1 } \}, \text{ and}$$
$$B_n = \{ \text{ compositions in } S_n, \text{ whose last part is not 1 } \},$$

so that $a_n = |S_n|$ is the value we wish to enumerate, and

$$S_n = A_n \sqcup B_n, \qquad \text{so that } a_n = |A_n| + |B_n|.$$

Note that $a_0 = 0$, and $a_1 = |\{(1)\}| = 1$.
The function from $A_n$ which takes $(\alpha_1, \dots, \alpha_\ell, 1)$ to $(\alpha_1, \dots, \alpha_\ell)$ is a function from $A_n$ to $S_{n-1}$ since $(\alpha_1 + \cdots + \alpha_\ell) + 1 = n$ and $\alpha_i$ is odd for all $i$. Its inverse, appending a good composition of $n-1$ by 1, is well-defined, so $A_n$ is in bijection with $S_{n-1}$.
Similarly, the function from $B_n$ which takes $(\alpha_1, \dots, \alpha_\ell)$ to $(\alpha_1, \dots, \alpha_\ell - 2)$ is a function from $B_n$ to $S_{n-2}$, since $(\alpha_1 + \cdots + \alpha_\ell) + 2 = n$ and $\alpha_i$ is odd, and $\alpha_\ell \geq 3$ and odd so that $\alpha_\ell - 2 \geq 1$ and odd. Its inverse, adding 2 to the last part, is well-defined, so $B_n$ is in bijection with $S_{n-2}$.
Thus

$$a_n = |S_n| = |A_n| + |B_n| = |S_{n-1}| + |S_{n-2}| = a_{n-1} + a_{n-2}.$$

Since $a_0 = 0 = f_0$ and $a_1 = 1 = f_1$, we have $\boxed{a_n = f_n}$.

(v) The number of sequences $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ of 0s and 1s such that $\varepsilon_1 \leq \varepsilon_2 \geq \varepsilon_3 \leq \varepsilon_4 \geq \cdots$.

*Solution:* Let

$$S_n = \{ \text{ good sequences } \},$$
$$A_n = \{ \text{ sequences in } S_n, \text{ whose value is 0 } \}, \text{ and}$$
$$B_n = \{ \text{ compositions in } S_n, \text{ whose value is 1 } \},$$

so that $a_n = |S_n|$ is the value we wish to enumerate, and

$$S_n = A_n \sqcup B_n, \qquad \text{so that } a_n = |A_n| + |B_n|.$$

Note that $a_1 = |\{(0), (1)\}| = 2$, and $a_2 = |\{(0,0), (0,1), (1,1)\}| = 3$.
We consider two cases:

(i) $\underline{n \text{ odd}}$, so that the last inequality is '$\geq$'.
If $\varepsilon \in A_n$, then $\varepsilon_{n-1} \geq \varepsilon_n = 0$, which puts no restriction on $\varepsilon_{n-1}$, nor any previous $\varepsilon_i$. So good sequences

$$\varepsilon_1 \leq \varepsilon_2 \geq \varepsilon_3 \leq \cdots \leq \varepsilon_{n-1} \geq 0$$

are in bijection with good sequences

$$\varepsilon_1 \leq \varepsilon_2 \geq \varepsilon_3 \leq \cdots \leq \varepsilon_{n-1},$$

i.e. $A_n$ is in bijection with $S_{n-1}$.
If $\varepsilon \in B_n$, then $\varepsilon_{n-1} \geq \varepsilon_n = 1$, so $\varepsilon_{n-1} = 1$. However, $\varepsilon_{n-2} \leq \varepsilon_{n-1} = 1$ puts no restriction on $\varepsilon_{n-2}$, nor any previous $\varepsilon_i$. So good sequences

$$\varepsilon_1 \leq \varepsilon_2 \geq \varepsilon_3 \leq \cdots \geq \varepsilon_{n-2} \leq 1 \geq 1$$

are in bijection with good sequences

$$\varepsilon_1 \leq \varepsilon_2 \geq \varepsilon_3 \leq \cdots \leq \varepsilon_{n-2},$$

i.e. $B_n$ is in bijection with $S_{n-2}$.

(ii) $\underline{n \text{ even}}$, so that the last inequality is '$\leq$'.
This follows exactly as in the previous case, except now $A_n$ is in bijection with $S_{n-2}$ and $B_n$ is in bijection with $S_{n-1}$.

Either way,

$$a_n = |S_n| = |A_n| + |B_n| = |S_{n-1}| + |S_{n-2}| = a_{n-1} + a_{n-2}.$$

Since $a_1 = 2 = f_3$ and $a_2 = 3 = f_4$, we have $\boxed{a_n = f_{n+2}}$.

(vi) The number of sequences $(T_1, T_2, \ldots, T_k)$ of subsets $T_i$ of $[n]$ such that $T_1 \subseteq T_2 \supseteq T_3 \subseteq T_4 \supseteq \cdots$.

*Solution:* For each $i \in [n]$, assign a sequence $\varepsilon^{(i)} = (\varepsilon_1^{(i)}, \ldots, \varepsilon_k^{(i)})$ by

$$\varepsilon_j^{(i)} = \begin{cases} 0 & \text{if } i \notin T_j, \\ 1 & \text{if } i \in T_j. \end{cases}$$

Note that

$$T_j \subseteq T_{j+1} \text{ if and only if } \varepsilon_j^{(i)} \leq \varepsilon_{j+1}^{(i)} \text{ for all } i \in [n],$$

and

$$T_j \supseteq T_{j+1} \text{ if and only if } \varepsilon_j^{(i)} \geq \varepsilon_{j+1}^{(i)} \text{ for all } i \in [n].$$

So by spanning over $i \in [n]$, we get a sequence $\varepsilon = (\varepsilon^{(1)}, \ldots, \varepsilon^{(k)})$ of sequences $\varepsilon^{(i)} = (\varepsilon_1^{(i)}, \ldots, \varepsilon_k)$ of 1's and 0's satisfying

$$\varepsilon_1^{(i)} \leq \varepsilon_2^{(i)} \geq \varepsilon_3^{(i)} \leq \varepsilon_4^{(i)} \geq \cdots.$$

Similarly, any such a sequence of good sequences gives a sequence of subset $T_i$ of $[n]$ such that $T_1 \subseteq T_2 \supseteq T_3 \subseteq T_4 \supseteq \cdots$. Since the sequences corresponding to each $i$ are independent of each other, and the number of good sequences $\varepsilon^{(i)}$ is $f_{k+2}$ by the previous part, we have the number of good sequences of subsets is given by

$$|\{\varepsilon^{(1)}\}| \cdots |\{\varepsilon^{(n)}\}| = f_{k+2} \cdots f_{k+2} = \boxed{f_{k+2}^n}.$$

(vii) The sum $\sum \alpha_1\alpha_2\cdots\alpha_\ell$ over all $2^{n-1}$ compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of $n$.
[Hint: this sum counts the number of ways of inserting at most one vertical bar in each of the $n-1$ spaces between stars in a line of $n$ stars, and then circling one star in each compartment. Now try replacing bars, un-circled stars, and circled stars by 1's, 2's, and 1's, respectively. Use a previous part.]

*Solution:* Let $S_n$ be the set of strong stars and bars arrangements with $n$ stars and $\ell-1$ bars, together with a choice of distinguished star for each of the $\ell$ regions. For example, if $n=10$ and $\ell=3$, elements of $S_n$ include

$$\circledast|\circledast *|* * * * * * *, \quad \circledast|*\circledast|* * * * * * *, \quad \text{and} \quad *\circledast * *|* * \circledast|\circledast * *.$$

Since strong star and bar arrangements with $n$ stars and $\ell-1$ bars are in bijection with compositions of $n$ with $\ell$ parts, and for each such arrangement corresponding to the composition $(a_1, \ldots, a_\ell)$, there are $a_1\cdots a_\ell$ ways to choose the distinguished stars (product rule), we have

$$|S_n| = \sum_{\text{comps } \alpha \text{ of } n} \alpha_1\cdots\alpha_\ell.$$

Now for each arrangement in $S_n$ assign the sequence of 1's and $2's$ by replacing bars, uncircled stars, and circled stars by 1's, 2's, and 1's, respectively. For example, the sequences corresponding to the arrangements above are

$$111212222222, \quad 112112222222, \quad \text{and} \quad 212212211122,$$

respectively. For a fixed $\ell$, there are $\ell-1$ bars, $\ell$ circled stars, and $n-\ell$ uncircled stars, so the values sum to $(\ell-1)*1+\ell*1+(n-\ell)*2 = \ell-1+\ell+2n-2\ell = 2n-1$, which is independent of $\ell$. So we have mapped $S_n$ to compositions of $2n-1$ whose parts are all 1's and 2's.
Now, lets consider the inverse operation. Since $2n-1$ is odd, any composition of $2n-1$ whose parts are all 1's and 2's will have an odd number of 1's. So for any such composition, replace all the $2's$ by stars, and then replace each of the 1's from left to right, alternating, with circled stars and bars (the first 1 becomes a circled star, the second 1 becomes a bar, and so on). If there are $r$ 2's, then there are $s = 2n-1-2r$ 1's; and $(s+1)/2$ of those 1's will become circled stars. So the subsequent arrangement will consist of

$$r+(s+1)/2 = r+(2n-1-2r+1)/2 = r+n-r = n \qquad \text{stars (circled and uncircled)},$$

and

$$(s-1)/2 = (2n-1-2r-1)/2 = n-r-1 \qquad \text{bars.}$$

[Note that since, over all such sequences, $r$ can range from 0 to $n-1$, the number of bars can range from 0 to $n-1$, as desired] So this inverse from all compositions of $2n-1$ whose parts are 1's and 2's is a well-defined map to $S_n$. Therefore, we have a bijection.
Thus, by part (a)(iii),

$$\sum_{\text{comps } \alpha \text{ of } n} \alpha_1\cdots\alpha_\ell = |S_n| = f_{2n-1}.$$

(b) Consider the identity

$$f_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}.$$

(i) Check this identity for $n = 2$ and 3.

*Solution:*

$$f_2 = \sum_{k=0}^{1} \binom{1-k}{k} = \binom{1}{0} + \binom{0}{1} = 1 + 0 = 1\checkmark$$

$$f_3 = \sum_{k=0}^{2} \binom{1-k}{k} = \binom{2}{0} + \binom{1}{1} + \binom{0}{2} = 1 + 1 + 0 = 2\checkmark$$

(ii) Prove this identity recursively by showing that it satisfies the Fibonacci recurrence, and that it holds for the first 2 values.

*Solution:* Recall the identity that

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1},$$

and let $F(n+1) = \sum_{k=0}^{n} \binom{n-k}{k}$. Then

$$\begin{aligned}
F(n) + F(n-1) &= \sum_{k=0}^{n-1} \binom{n-k-1}{k} + \sum_{k=0}^{n-2} \binom{n-k-2}{k} \\
&= \sum_{k=0}^{n-1} \binom{n-k-1}{k} + \sum_{\ell=1}^{n-1} \binom{n-\ell-1}{\ell-1} \text{ (setting } \ell = k+1) \\
&= \binom{n-1}{0} + \sum_{k=1}^{n-1} \left( \binom{n-k-1}{k} + \binom{n-k-1}{k-1} \right) \\
&= \binom{n}{0} + \sum_{k=1}^{n-1} \binom{n-k}{k} \\
&= \sum_{k=0}^{n} \binom{n-k}{k},
\end{aligned}$$

since $\binom{n-n}{n} = 0$. But the last line is just $F(n+1)$. So $F(n)$ satisfies the Fibonacci recurrence and $F(2) = f_2$ and $F(3) = f_3$. Thus $F(n) = f_n$.

(iii) Prove this identity combinatorially. Namely, first show combinatorially that the number of $k$-subsets of $[n-1]$ containing no two consecutive integers is $\binom{n-k}{k}$, and then use (a)(i).

*Solution:* Let $S_{n,k}$ be $k$-subsets of $[n-1]$ containing no two consecutive integers. For $A = \{a_1 < a_2 < \cdots < a_k\} \in S_{n,k}$, consider the map

$$\varphi : \{a_1, a_2, \ldots, a_k\} \mapsto \{a_1, a_2 - 1, a_3 - 2, \ldots, a_k - k + 1\} = \{b_1, b_2, b_3, \ldots, b_k\},$$

i.e. $b_i = a_i - i + 1$. Since $A$ contains no two consecutive integers,

$$b_{i+1} - b_i = a_{i+1} - (i+1) + 1 - (a_i - i + 1) = a_{i+1} - a_i - 1 > 0,$$

so this map is a well-defined map of sets. Moreover, $b_1 = a_1 \geq 1$ and $b_k = a_k - k + 1 \leq n - 1 - k + 1 = n - k$, and $b_1 < b_i < b_k$, so $\varphi(A) \in \binom{[n-k]}{k}$.
The inverse, sending a set $B = \{b_1 < b_2 < \cdots < b_k\} \in \binom{[n-k]}{k}$ to

$$\varphi^{-1} : \{b_1, b_2, \ldots, b_k\} \mapsto \{b_1, b_2 + 1, b_3 + 2, \ldots, b_k + k - 1\}$$

is well-defined with image in $S_{n,k}$, since we have dilated the entries of $B$ (giving no two consecutive elements) and $b_k + k - 1 \leq n - k + k - 1 = n - 1$. So $S_{n,k}$ is in bijection with $\binom{[n-k]}{k}$.
Thus by (a)(i),

$$\sum_{k=0}^{n} \binom{n-k}{k} = \sum_{k=0}^{n-1} \binom{n-k}{k} = \sum_{k=0}^{n-1} |S_{n,k}| = f_{n-1+2} = f_{n+1}.$$

(c) Note that EC1, Example 1.1.12 to computes the generating function for the sequence $(a_i)_{i\in\mathbb{N}}$, where $a_i = f_{i+1}$ (so $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3$, and so on). Repeat this computation for $(f_i)_{i\in\mathbb{Z}_{>0}}$, making the appropriate changes to accommodate the shift.

*Solution:* Just do it.