Solutions for HW3

Exercise 9.

(a) (i) For each composition α of 3, give S_α.

Solution:

$$S_{(3)} = \emptyset$$
$$S_{(2,1)} = \{2\}$$
$$S_{(1,2)} = \{1\}$$
$$S_{(1,1,1)} = \{1, 2\}$$

(ii) Prove that the map $\theta : \alpha \to S_\alpha$ is a bijection between ℓ-compositions of n and $(\ell - 1)$-subsets of $[n-1]$.
[Hint: define θ^{-1} and show that it is well-defined.]

Solution: First, we see that θ is well-defined by checking that the image of every ℓ-composition α of n is a subset of $[n-1]$. Since every term in α is a positive integer, the sequence $\alpha_1, \alpha_1 + \alpha_2, \ldots$ is strictly increasing, so S_α is a set of positive integers. Since the terms of α sum to n and are positive, every partial sum is strictly between 0 and n, so $S_\alpha \subseteq [n-1]$.
Next, for any $S \subset [n-1]$, linearly order its elements $x_1 < x_2 < \cdots < x_{\ell-1}$. Then I claim that

$$\begin{aligned} \theta^{-1} : \binom{[n-1]}{\ell-1} &\to \{\ \ell\text{-compositions of } n\ \} \\ \{x_1, \ldots, x_{\ell-1}\} &\mapsto \alpha = (\alpha_1, \ldots, \alpha_\ell) \\ &\qquad = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_{\ell-1} - x_{\ell-2}, n - x_{\ell-1}), \end{aligned}$$

and that θ^{-1} is well-defined on $2^{[n-1]}$. First, since the $x_i's$ are linearly ordered positive integers, each of the α_i's are also positive integers. Next

$$\sum_{i=1}^{\ell} \alpha_i = x_1 + \sum_{i=2}^{\ell-1}(x_i - x_{i-1}) + n - x_{\ell-1} = x_1 + x_{\ell-1} - x_1 + n - x_{\ell-1} = n,$$

so α is a composition of n. Thus θ^{-1} is well-defined. Finally, since θ produces elements of S_α in their linear order,

$$\begin{aligned} \theta^{-1}(\theta(\alpha)) &= \theta^{-1}(\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1}\}) \\ &= (\alpha_1, \alpha_1 + \alpha_2 - \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 - (\alpha_1 + \alpha_2), \ldots, \\ &\qquad \alpha_1 + \cdots + \alpha_{\ell-1} - (\alpha_1 + \cdots + \alpha_{\ell-2}), n - (\alpha_1 + \cdots + \alpha_{\ell-1})) \\ &= (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell), \end{aligned}$$

so we have defined the inverse function correctly.
Since θ is well-defined and invertible, it is a bijection.

(b) Describe a bijection between weak ℓ-compositions of n and arrangements of stars and bars, and conclude how many weak ℓ-compositions there are of n.

Solution: For each weak ℓ-composition α of n, assign the arrangement of stars and bars given by α_1 stars, followed by a bar, then α_2 stars followed by a bar, and so on, concluding in α_ℓ stars. Since the terms of α sum to n, there will be n stars. Since the last part doesn't get a bar, there will be $\ell - 1$ bars. This process is invertible, since the parts of α can be read off of any linear arrangement of n stars and $\ell - 1$ bars. Since we are concerned with weak compositions of n, there is no restriction on the proximity of the bars for the inverse to be well-defined.

Thus weak ℓ-compositions of n are in bijection with rearrangements of n stars and $\ell - 1$ bars, of which there are

$$\binom{n + (\ell - 1)}{n} = \binom{n + (\ell - 1)}{\ell - 1}.$$

(c) Give a bijection between E_n, the set of compositions of n with an even number of even parts, and O_n, the set of compositions of n with an odd number of even parts.
[For example, $E_3 = \{(1, 1, 1), (3)\}$ and $O_3 = \{(2, 1), (1, 2)\}$.]

Solution: View compositions in the form of their star and bar arrangements. Consider the map ι from compositions to compositions given by toggling the last bar (removing it if it's there, and adding it if it is not). For example

$$* * *| * | * *| * ** \overset{\iota}{\longleftrightarrow} * * *| * | * *| * *|*$$
$$(3, 1, 2, 3) \overset{\iota}{\longleftrightarrow} (3, 1, 2, 2, 1).$$

It is straightforward to see that ι is an involution on the set of compositions of n (it is a bijection from the set to itself, which is its own inverse). Now let's see what happens to the parity of even parts of a composition under ι.

If ι adds a bar, it splits α_ℓ into $\alpha_\ell - 1$ and 1.

If α_ℓ is odd, ι replaces one odd part with an even part and an odd part.

If α_ℓ is even, ι replaces one even part with an two odd parts.

If ι removes a bar, it combines $\alpha_{\ell-1}$ and $\alpha_\ell = 1$ into $\alpha_{\ell-1} + 1$.

If $\alpha_{\ell-1}$ is odd, ι replaces two odd parts with one even part.

If $\alpha_{\ell-1}$ is even, ι replaces an even part and an odd part with one odd part.

In all four cases, ι toggles the parity of the number of even parts in α, and so restricts to a bijection between E_n and O_n.

Use your bijection to conclude how many compositions there are with an even number of even parts.

Solution: Since every composition of n is either in E_n or in O_n,

$$2^{n-1} = |\{ \text{ compositions of } n \ \}| = |E_n \sqcup O_n| = |E_n| + |O_n| = 2|E_n|,$$

we have $|E_n| = 2^{n-2}$.

(d) Show that the total number of all parts in all compositions of n is $(n + 1)2^{n-2}$.
[For example, the compositions of 3 are $(3), (2, 1), (1, 2)$, and $(1, 1, 1)$, which all together have 8 parts; and $8 = (3 + 1)2^{3-2}$.]
[Hint: Use a stars and bars argument: if you line up all the compositions, how many bars appear in total? Explain why the total number of parts is equal to the total number of bars, plus the total number of compositions.]

Solution: Drawing all compositions in their stars and bars representation, note that

the first bar is in exactly half of the compositions,

the second bar is in exactly half of the compositions, and so on. Thus there are a total of $\frac{1}{2}2^{n-1} + \cdots + \frac{1}{2}2^{n-1} = (n-1)2^{n-2}$ bars appearing in the set of all compositions of n. But in every composition, there is exactly one more part than there are bars. So the total number of parts is

$$(n-1)2^{n-2} + 2^{n-1} = (n-1+2)2^{n-2} = (n+1)2^{n-2}.$$

Exercise 10.

(a) Explain why there are $\binom{n-1}{\ell}$ positive integer solutions to

$$x_1 + x_2 + \cdots + x_\ell < n$$

and $\binom{n+\ell}{\ell}$ non-negative integer solutions to

$$x_1 + x_2 + \cdots + x_\ell \leq n$$

by setting up a linear equations that have the appropriate number of solutions.

Solution: The positive integer solutions to $x_1 + x_2 + \cdots + x_\ell < n$ are in bijection with the positive integer solutions to

$$x_1 + x_2 + \cdots + x_\ell + y = n$$

(where $y = n - (x_1 + x_2 + \cdots + x_\ell)$). Thus there are $\binom{n-1}{\ell-1+1} = \binom{n-1}{\ell}$ solutions.

Similarly, nonnegative integer solutions to $x_1 + x_2 + \cdots + x_\ell \leq n$ are in bijection with the nonnegative integer solutions to

$$x_1 + x_2 + \cdots + x_\ell + y = n$$

(where again $y = n - (x_1 + x_2 + \cdots + x_\ell)$). Thus there are $\binom{n+\ell+1-1}{\ell-1+1} = \binom{n+\ell}{\ell}$ solutions.

(b) (i) How many solutions are there to the equation

$$x_1 + x_2 + x_3 = 10,$$

where x_1, x_2, and x_3 are nonnegative integers? How many solutions are there if x_1, x_2, and x_3 are positive integers?

Solution: There are $\binom{10+3-1}{3-1} = \binom{12}{2}$ nonnegative integer solutions and $\binom{10-1}{3-1} = \binom{9}{2}$ positive integer solutions.

(ii) How many solutions are there to the equation

$$x_1 + x_2 + x_3 \leq 10,$$

where x_1, x_2, and x_3 are nonnegative integers?

Solution: By part (a), are $\binom{10+3}{3} = \binom{13}{3}$ nonnegative integer solutions.

Exercise 11.

(a) Suppose you've got eight varieties of doughnuts to choose from at a doughnuts shop.

(i) How many ways can you pick 6 doughnuts?

Solution: Six indistinguishable choices with eight distinguishable categories has $\binom{6+8-1}{8-1} = \binom{13}{7}$ possible outcomes.

(ii) How many ways can you pick a dozen doughnuts?

Solution: Twelve indistinguishable choices with eight distinguishable categories has $\binom{12+8-1}{8-1} = \binom{19}{7}$ possible outcomes.

(iii) How many ways can you pick a dozen doughnuts with at least one of each kind?

Solution: Twelve indistinguishable choices with eight distinguishable categories, with at least once choice allocated to each category has $\binom{12-1}{8-1} = \binom{11}{7}$ possible outcomes.

(b) How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a jar contain if it has 20 coins in it?

Solution: Twenty indistinguishable choices with five distinguishable categories has $\binom{20+5-1}{5-1} = \binom{24}{4}$ possible outcomes.

Exercise 12.

(a) List the 3-multisets on [2].

Solution: $M = ([2], \nu)$ with

$$\nu: \begin{array}{l} 1 \mapsto 3 \\ 2 \mapsto 0 \end{array} \qquad \nu: \begin{array}{l} 1 \mapsto 2 \\ 2 \mapsto 1 \end{array} \qquad \nu: \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 2 \end{array} \qquad \nu: \begin{array}{l} 1 \mapsto 0 \\ 2 \mapsto 3 \end{array}$$

(b) How many 5-multisets are there on [8]?

Solution: The 5-multisets of [8] are in bijection with weak 8-compositions of 5, so there are $\binom{5+8-1}{8-1} = \binom{12}{7}$.

(c) How many multisets are there on [8]? (Of any size)

Solution: There are infinitely many multisets on and non-empty set.

(d) Describe a bijection between k-multisets on $[n]$ and stars and bars arrangements with k stars and $n-1$ bars.

Solution: To a k-multiset $M = ([n], \nu)$, assign the stars and bars arrangement given by $\nu(1)$ stars followed by a bar, then $\nu(2)$ stars followed by a bar, and so on, concluding in $\nu(n)$ stars. Since $\sum_{x \in [n]} \nu(x) = k$, there are k stars in total. Since there is one fewer bar than elements of $[n]$, there are $n-1$ stars. Since the values of $\nu(i)$ can be read off sequentially from any stars and bars arrangement, this process is invertible. Every stars and bars arrangement of the right size will put out a sequence of n numbers in $\mathbb{N}$, so the inverse is well-defined. Therefore, we have described a bijection.