Solutions for HW2

Exercise 4. (a) Give both the generating and exponential generating functions for

$$f(n) = 3^n; \qquad g(n) = 3; \qquad \varphi(n) = 3n; \qquad \text{and} \qquad \psi(n) = n!3^n; \qquad \text{for } n \in \mathbb{N}.$$

For each, give your answer in series form. Whenever possible, also give your answer in closed form.

Solution:

$$\sum_{n\in\mathbb{N}} f(n)x^n = \sum_{n\in\mathbb{N}} 3^n x^n = \frac{1}{1-3x}$$
$$\sum_{n\in\mathbb{N}} f(n)\frac{x^n}{n!} = \sum_{n\in\mathbb{N}} 3^n \frac{x^n}{n!} = e^{3x}$$

$$\sum_{n\in\mathbb{N}} g(n)x^n = \sum_{n\in\mathbb{N}} 3x^n = \frac{3}{1-x}$$
$$\sum_{n\in\mathbb{N}} g(n)\frac{x^n}{n!} = \sum_{n\in\mathbb{N}} 3\frac{x^n}{n!} = 3e^x$$

$$\sum_{n\in\mathbb{N}} \varphi(n)x^n = \sum_{n\in\mathbb{N}} 3nx^n = 3x\sum_{m\in\mathbb{N}} (m+1)x^m = \frac{3x}{(1-x)^2}$$
$$\sum_{n\in\mathbb{N}} \varphi(n)\frac{x^n}{n!} = \sum_{n\in\mathbb{N}} 3n\frac{x^n}{n!} = 3x\sum_{m\in\mathbb{N}} \frac{x^m}{m!} = 3xe^x$$

$$\sum_{n\in\mathbb{N}} \psi(n)x^n = \sum_{n\in\mathbb{N}} n!3^n x^n \text{ (no closed form)}$$
$$\sum_{n\in\mathbb{N}} \psi(n)\frac{x^n}{n!} = \sum_{n\in\mathbb{N}} n!3^n\frac{x^n}{n!} = \frac{1}{1-3x}$$

(b) Verify the rule for multiplying basic and exponential formal series for $I = \mathbb{N}$, for the first 4 coefficients. In other words, calculate c_n for $n = 0, 1, 2, 3$ by multiplying out the left hand side of

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots,$$

and comparing coefficients (and similarly for the exponential case).

Solution: Generating function rule:

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots)$$
$$= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2$$
$$+ (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \cdots$$

$$a_0b_0 = \sum_{i=0}^{0} a_ib_{0-i}\checkmark$$

$$a_0b_1 + a_1b_0 = \sum_{i=0}^{1} a_ib_{1-i}\checkmark$$

$$a_0b_2 + a_1b_1 + a_2b_0 = \sum_{i=0}^{2} a_ib_{2-i}\checkmark$$

$$a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 = \sum_{i=0}^{3} a_ib_{3-i}\checkmark$$

Exponential generating function rule:

$$(a_0 + a_1x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots)(b_0 + b_1x + b_2\frac{x^2}{2!} + b_3\frac{x^3}{3!} + \cdots)$$
$$= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + 2a_1b_1 + a_2b_0)\frac{x^2}{2!}$$
$$+ (a_0b_3 + 3a_1b_2 + 3a_2b_1 + a_3b_0)\frac{x^3}{3!} + \cdots$$

$$a_0b_0 = \sum_{i=0}^{0} a_ib_{0-i}\checkmark$$

$$a_0b_1 + a_1b_0 = \sum_{i=0}^{1} \binom{1}{i} a_ib_{1-i}\checkmark$$

$$a_0b_2 + 2a_1b_1 + a_2b_0 = \sum_{i=0}^{2} \binom{2}{i} a_ib_{2-i}\checkmark$$

$$a_0b_3 + 3a_1b_2 + 3a_2b_1 + a_3b_0 = \sum_{i=0}^{3} \binom{3}{i} a_ib_{3-i}\checkmark$$

(c) Verify that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

by solving for b_n in the equation

$$\left(\sum_{n=0}^{\infty} a_nx^n\right)\left(\sum_{n=0}^{\infty} b_nx^n\right) = \sum_{n=0}^{\infty} c_nx^n,$$

where

$$a_0 = a_1 = c_0 = 1, \text{ and } a_n, c_n = 0 \text{ otherwise},$$

i.e., $\sum_{n=0}^{\infty} a_n x^n = 1 + x$ and $\sum_{n=0}^{\infty} c_n x^n = 0$. [See EC1, Example 1.1.5; but be more explicit.]

Solution: Write $\frac{1}{1-x} = \sum_{n=0}^{\infty} b_n x^n$, so that

$$\begin{aligned}
1 &= (1-x)\left(\frac{1}{1-x}\right) \\
&= (1-x)(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots) \\
&= b_0 + (b_1 - b_0)x + (b_2 - b_1)x^2 + (b_3 - b_2)x^3 + \cdots \\
&= b_0 + \sum_{n=1}^{\infty} (b_n - b_{n-1}) x^n.
\end{aligned}$$

Comparing coefficients on either side, we have

$$b_0 = 1 \qquad \text{and} \qquad b_n - b_{n-1} = 0 \text{ for } n > 0;$$

i.e.

$$b_0 = 1 \qquad \text{and} \qquad b_n = b_{n-1} \text{ for } n > 0.$$

Thus $b_n = 1$ for $n \in \mathbb{N}$, i.e. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ as expected.

Exercise 5. For each of the following identities,

(i) check by hand for $n = 3$;
(ii) verify using the binomial theorem, evaluating for specific values of x;
(iii) give a combinatorial proof of the identity.

(a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$;

Solution:

(i) check by hand for $n = 3$:

$$\sum_{k=0}^{3} \binom{3}{k} = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8 = 2^3 \checkmark$$

(ii) verify using the binomial theorem, evaluating for specific values of x:
Evaluate the binomial theorem at $x = 1$ to get

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k = (1+1)^n = 2^n.$$

(iii) give a combinatorial proof of the identity:
Let $f(n)$ be the number of subsets of $[n]$. On the one hand, we know $f(n) = |2^{[n]}| = 2^n$. On the other hand, one can sum up the number of subsets of size k for $k = 0, 1, \ldots, n$, which gives

$$f(n) = \sum_{k=0}^{n} \left| \binom{[n]}{k} \right| = \sum_{k=0}^{n} \binom{n}{k}.$$

Thus

$$\sum_{k=0}^{n} \binom{n}{k} = f(n) = 2^n.$$

(b) $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ for $n > 0$

[Hint: for (ii), differentiate first].

Solution:

(i) check by hand for $n = 3$:

$$\sum_{k=0}^{3} k\binom{3}{k} = 0 + \binom{3}{1} + 2\binom{3}{2} + 3\binom{3}{3} = 3 + 6 + 3 = 12 = 3 * 2^{3-1}\checkmark.$$

(ii) verify using the binomial theorem, evaluating for specific values of x:
Differentiate the binomial theorem to get

$$\sum_{k=0}^{n} \binom{n}{k} kx^{k-1} = n(1+x)^{n-1}.$$

Then evaluate at $x = 1$ to get

$$\sum_{k=0}^{n} \binom{n}{k} k = n(1+1)^{n-1} = n2^{n-1}.$$

(iii) give a combinatorial proof of the identity:
Let $f(n)$ be the number of ways to pick a committee of any size from n people, together with a person to run meetings. On the one hand, you can choose a committee of k people (for k between 1 and n) and then choose the leader, giving

$$f(n) = \sum_{k=1}^{n} \binom{n}{k} k = \sum_{k=0}^{n} \binom{n}{k} k,$$

by the sum and product rules. On the other hand, you can choose the leader $x \in [n]$ first, and then a committee from the remaining set $[n] - \{x\}$; giving

$$f(n) = n2^{|[n]|-1} = n2^{n-1}.$$

Thus

$$\sum_{k=0}^{n} \binom{n}{k} k = f(n) = n2^{n-1}.$$

(c) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ for $n > 0$

[Hint: For (iii), rewrite the identity by moving the negative terms to the right (generally good practice for combinatorial proofs). Construct a bijection between the set E_n of all subsets of $[n]$ that have an even number of elements and O_n, the odd counterpart.].

Solution:

(i) check by hand for $n = 3$:

$$\sum_{k=0}^{3} (-1)^k \binom{3}{k} = \binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 1 - 3 + 3 - 1 = 0\checkmark$$

(ii) verify using the binomial theorem, evaluating for specific values of x:
Evaluate the binomial theorem at $x = -1$ to get

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1-1)^n = 0.$$

(iii) give a combinatorial proof of the identity:
The identity of interest is equivalent to the identity

$$\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k}.$$

Let E_n be the set of subsets of $[n]$ that have an even number of elements, and O_n be the set of subsets of $[n]$ that have an odd number of elements. Note that

$$|E_n| = \sum_{k \text{ even}} \binom{n}{k} \quad \text{and} \quad |O_n| = \sum_{k \text{ odd}} \binom{n}{k}.$$

Define a map on E_n by

$$\varphi(A) = \begin{cases} A - \{n\} & \text{if } n \in A, \\ A \sqcup \{n\} & \text{if } n \notin A. \end{cases}$$

Since adding or removing one element changes the parity of the set A, $\varphi : E_n \to O_n$. The same map on O_n is exactly the inverse function $\varphi^{-1} : O_n \to E_n$ (i.e. φ is a fixed-point free involution on $2^{[n]}$ that swaps E_n and O_n). Therefore

$$\sum_{k \text{ even}} \binom{n}{k} = |E_n| = |O_n| = \sum_{k \text{ odd}} \binom{n}{k},$$

giving the expected identity.

Exercise 6. Use the multiplication rules for exponential series to show that

$$1 = e^x e^{-x} \qquad \text{implies} \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \quad \text{for } n > 0$$

(our third proof, making EC1, Example 1.1.6 more explicit).

Solution: Write

$$e^{-x} = \sum_{n\in\mathbb{N}} a_n \frac{x^n}{n!} \qquad \text{and} \qquad e^x = \sum_{n\in\mathbb{N}} b_n \frac{x^n}{n!},$$

where

$$a_n = (-1)^n \qquad \text{and} \qquad b_n = 1 \qquad \text{for } n \in \mathbb{N}.$$

Thus since

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k,$$

we have

$$1 = e^{-x}e^x = \left(\sum_{n\in\mathbb{N}} (-1)^n \frac{x^n}{n!}\right)\left(\sum_{n\in\mathbb{N}} \frac{x^n}{n!}\right) = \sum_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k\right) x^n.$$

Comparing coefficients on either side, we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0 \qquad \text{for all } n \in \mathbb{Z}_{>0}.$$

Exercise 7. Use $\frac{1}{1-x} = \sum_{n\in\mathbb{N}} x^n$ and $e^x = \sum_{n\in\mathbb{N}} \frac{x^n}{n!}$ as *definitions* of $\frac{1}{1-x}$ and e^x, i.e.

$$\frac{1}{1-e^x} \text{ is short-hand for } F(G(x)), \text{ where } \begin{array}{l} F(x) = \sum_{n\in\mathbb{N}} x^n, \text{ and} \\ G(x) = \sum_{n\in\mathbb{N}} \frac{x^n}{n!}. \end{array}$$

Which of the following expressions are well-defined formal power series? Why? For those expressions that are well-defined, give their first few terms.

(i) e^{x+1} Answer: Here, $G(x) = 1 + x$ has non-zero linear term. So e^{x+1} is not well-defined.

(ii) e^{x+3x^2} Answer: Here, $G(x) = x + 3x^2$ has zero linear term. So e^{x+3x^2} is well-defined. Namely,

$$\begin{aligned} e^{x+3x^2} &= \sum_{n\in\mathbb{N}} \frac{(x+3x^2)^n}{n!} \\ &= \frac{(x+3x^2)^0}{0!} + \frac{(x+3x^2)^1}{1!} + \frac{(x+3x^2)^2}{2!} + \frac{(x+3x^2)^3}{3!} + \cdots \\ &= 1 + (x+3x^2) + \frac{1}{2}(x^2 + 6x^3 + 9x^4) + \frac{1}{3!}(x^3 + 9x^4 + 27x^5 + 27x^6) + \cdots \\ &= 1 + x + (3 + \frac{1}{2})x^2 + (3 + \frac{1}{3!})x^3 + \cdots . \end{aligned}$$

(iii) e^{e^x} Answer: Here, $G(x) = e^x$ has non-zero linear term. So e^{e^x} is not well-defined.

(iv) e^{e^x-1} Answer: Here, $G(x) = e^x - 1 = \sum_{k=1}^{\infty} x^k/k!$ has zero linear term. So e^{e^x-1} is well-defined. Namely,

$$\begin{aligned}
e^{e^x-1} &= \sum_{n\in\mathbb{N}} \frac{(x + x^2/2 + x^3/3! + \cdots)^n}{n!} \\
&= 1 + (x + x^2/2 + x^3/3! + \cdots) + \frac{1}{2}(x + x^2/2 + x^3/3! + \cdots)^2 \\
&\quad + \frac{1}{3!}(x + x^2/2 + x^3/3! + \cdots) + \cdots \\
&= 1 + (x + x^2/2 + x^3/3! + \cdots) + \frac{1}{2}(x^2 + x^3 + \frac{7}{12}x^4 + \cdots) \\
&\quad + \frac{1}{3!}(x^3 + \frac{3}{2}x^4 + \frac{5}{4}x^5 + \cdots) + \cdots \\
&= 1 + x + x^2 + \frac{4}{3!}x^3 + \cdots .
\end{aligned}$$

(v) $\frac{1}{1-xe^x}$ Answer: Here, $G(x) = xe^x = \sum_{k=1}^{\infty} x^k/(k-1)!$ has zero linear term. So $\frac{1}{1-xe^x}$ is well-defined. Namely,

$$\begin{aligned}
\frac{1}{1-xe^x} &= \sum_{n\in\mathbb{N}} (xe^x)^n = \sum_{n\in\mathbb{N}} x^n e^{nx} \\
&= 1 + x(1 + x + x^2/2 + x^3/3! + \cdots) + x^2(1 + 2x + 2^2x^2/2 + 2^3x^3/3! + \cdots) \\
&\quad + x^3(1 + 3x + 3^2x^2/2 + 3^3x^3/3! + \cdots) + \cdots \\
&= 1 + (x + x^2 + x^3/2 + x^4/3! + \cdots) + (x^2 + 2x^3 + 2x^4 + (2^3/3!)x^5 + \cdots) \\
&\quad + (x^3 + 3x^4 + (3^2/2)x^5 + (3^3/3!)x^6 + \cdots) + \cdots \\
&= 1 + x + 2x^2 + (7/2)x^3 + \cdots .
\end{aligned}$$

(vi) $\frac{1}{xe^x}$ Answer: Here, $G(x) = xe^x + 1$, which has non-zero linear term. So $\frac{1}{xe^x}$ is not well-defined.

Exercise 8. (EC1, exercise 1.8)

(a) Use the generalized binomial theorem to expand $\frac{1}{\sqrt{1-4x}}$ in series form.

$$\frac{1}{\sqrt{1-4x}} = (1 + (-4x))^{-1/2} = \sum_{n\in\mathbb{N}} \binom{-1/2}{n} x^n.$$

(b) Calculate $\frac{(2n)!}{n!}$ for $n = 1, 2, 3$. What is $\frac{(2n)!}{n!}$ in general?

$$\frac{(2*1)!}{1!} = 2$$
$$\frac{(2*2)!}{2!} = \frac{4*3*2*1}{2*1} = 2^2(3*1)$$
$$\frac{(2*3)!}{3!} = \frac{6*5*4*3*2*1}{3*2*1} = 2^3(5*3*1)$$

$$\frac{(2*n)!}{n!} = \frac{(2n)*(2n-1)*(2(n-1))*(2n-3)*\cdots*2*1}{n(n-1)\cdots*1} = 2^n(2n-1)(2n-3)\cdots 1.$$

Thus
$$(2n-1)(2n-3)\cdots 1 = \frac{(2n)!}{2^n n!}.$$

(c) Calculate $\binom{-1/2}{k}$ for $k = 1, 2, 3$ What is $\binom{1/2}{k}$ in general? [Note that you can factor $\frac{1}{2}$ from every term in the numerator. Then use part (b).]

Solution:
$$\binom{-1/2}{1} = \frac{1}{1!}(-1/2)$$
$$\binom{-1/2}{2} = \frac{1}{2!}(-1/2)(-1/2-1) = \frac{1}{2!}(-1/2)^2(1)(3)$$
$$\binom{-1/2}{3} = \frac{1}{3!}(-1/2)(-1/2-1)(-1/2-2) = \frac{1}{3!}(-1/2)^3(1)(3)(5)$$

$$\binom{-1/2}{k} = \frac{1}{k!}(-1/2)(-1/2-1)(-1/2-2)\cdots(1/2-(k-1)) = \frac{1}{k!}(-1/2)^k(1)(3)(5)\cdots(2k-1)$$
$$= \frac{1}{k!}(-1/2)^n\frac{(2k)!}{2^k k!} = (-1/4)^k\binom{2k}{k}.$$

(d) Conclude
$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n.$$

Solution: This is a direct combination of parts (a) and (c).

(e) Give a combinatorial proof of the identity $2\binom{2n-1}{n} = \binom{2n}{n}$.

Solution: Consider the set $\mathcal{S}_n$ of size-n subsets A of $[2n]$.

Claim: There is a fixed-point free involution on $\mathcal{S}_n$ given by
$$\varphi : A \mapsto [2n] - A.$$

Proof: Namely, for any $A \in \mathcal{S}$, since $|A| = n$, we have $|\varphi(A)| = n$, and so $\varphi(A) \in \mathcal{S}$ (i.e. $\varphi : \mathcal{S} \to \mathcal{S}$). Further, the complement of the complement of A in $[2n]$ is A itself, so φ is its own inverse (i.e. it is an involution). Finally, since $A \cap \varphi(A) = \emptyset$, we have $A \neq \varphi(A)$, i.e. φ is fixed-point free. $\square$

Further note that φ gives a bijection between $\mathcal{S}_1$, the set of size-n subsets of $[2n]$ containing $2n$, and $\mathcal{S}_0$, the set of size-n subsets of $[2n]$ not containing $2n$. This is because exactly one of A and $\varphi(A)$ contains n, for all $A \in \mathcal{S}$. Thus, since $|\mathcal{S}_1| = |\mathcal{S}_0|$, $\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_0$, we have

$$\binom{2n}{n} = |\mathcal{S}| = |\mathcal{S}_1 \sqcup \mathcal{S}_0| = |\mathcal{S}_1| + |\mathcal{S}_0| = 2|\mathcal{S}_0|.$$

But $\mathcal{S}_0$, the size-n subsets of $[2n]$ not containing n, are exactly the size-n subsets of $[2n-1]$. So $|\mathcal{S}_0| = \binom{2n-1}{n}$. Therefore

$$\binom{2n}{n} = |\mathcal{S}| = 2|\mathcal{S}_0| = 2\binom{2n-1}{n}.$$

(f) Find $\sum_{n=0}^{\infty} \binom{2n-1}{n} x^n$.

Solution: Using parts (e) and then (d), we have

$$\sum_{n=0}^{\infty} \binom{2n-1}{n} x^n = \sum_{n=0}^{\infty} \frac{1}{2}\binom{2n}{n} x^n = \frac{1}{2\sqrt{1-4x}}.$$