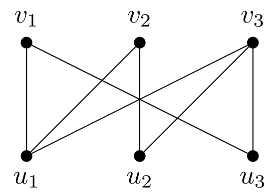
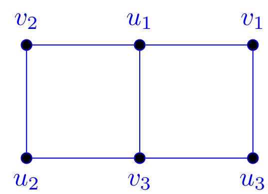
Exercise 34. (a) Use the recurrence relation $t(G) = t(G - e) + t(G/e)$ to count the number of spanning trees of



Remember to keep multiple edges!!

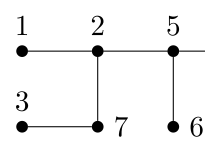
Solution: First redraw G so that none of the edges cross:



Recursing on $e = (v_3, u_1)$:

$$t(G) = t(G - e) + t(G/e) = 6 + 3 * 3.$$

(b) What is the Prüfer code for the following labeled tree?

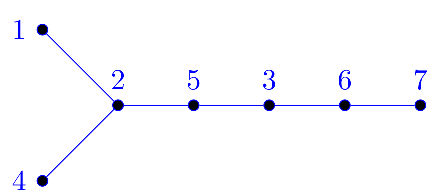


Check your answer by reversing the process and building the tree from the code.

Answer: 2,7,5,5,2.

(c) Draw the tree whose Prüfer code is $2, 2, 5, 3, 6$. Check your answer by calculating the Prüfer code that goes with your tree.

Solution:



(d) Draw a labeled K_3 (labeled with $1, 2, 3$), and list all the spanning trees, and the corresponding Prüfer code. Verify that there is a bijection between the labeled trees on 3 vertices and the length-1 Prüfer codes.

(e) How many spanning trees does K_7 have?

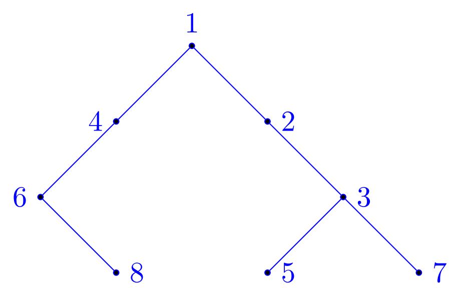
(f) How many labeled trees are there on 14 vertices?

Exercise 35. (a) For the following permutations w, draw $T(w)$ and $T'(w)$. Verify that $T(w)$ and $T'(w)$ are increasing.

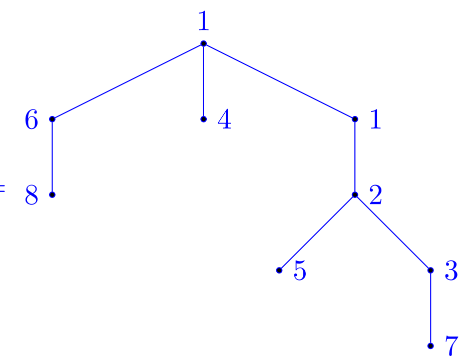
(i) $w = 68412537$

Solution:

$T(68412537) =$



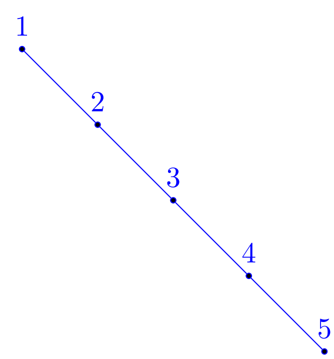
$T'(68412537) =$



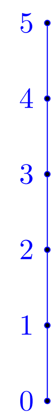
(ii) $w = 12345$

Solution:

$T(12345) =$



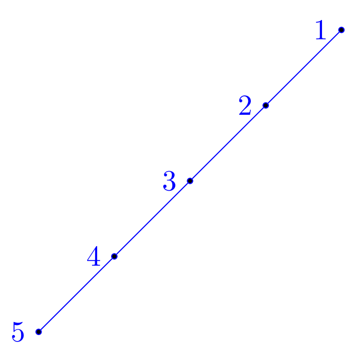
$T'(12345) =$



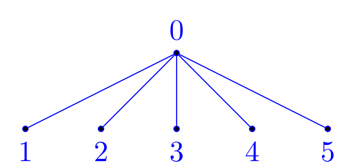
(iii) $w = 54321$

Solution:

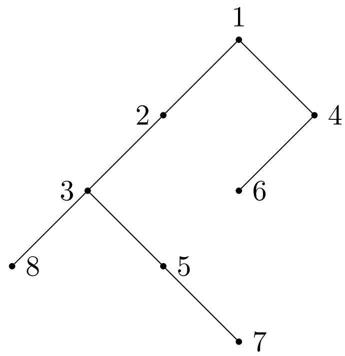
$T(54321) =$



$T'(54321) =$



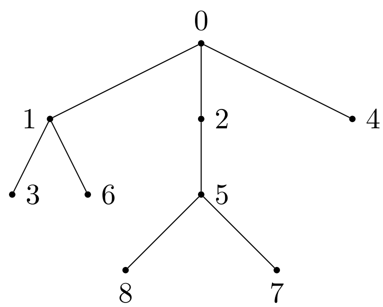
(b) Let

$T =$  .

If $T = T(w)$, what is w? Verify that the double rises, double descents, valleys, and peaks of w correspond to the correct behavior of successors.

Solution: If $T = T(w)$, then $w = 83572164$.

(c) Let

$T =$  .

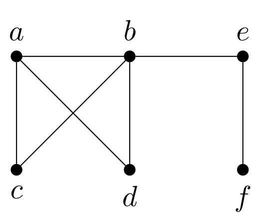
If $T = T'(w)$, what is w? Verify the following for $T'(w)$.

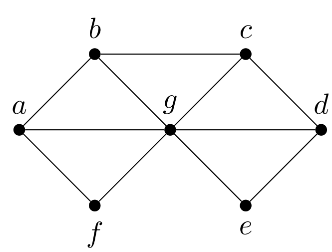
Solution: If $T = T'(w)$, then $w = 42587163$.

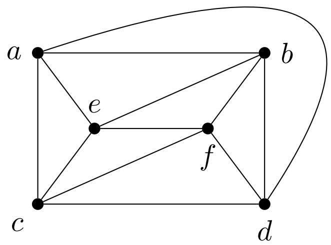
(i) The successors of 0 are just the left-to-right minima of w.
(ii) The leaves are $\{w(i) \mid i \in D(w) \cup \{n\}\}$. $\{w(i) \mid i \in D(w)\} = \{4, 8, 7, 6\}$

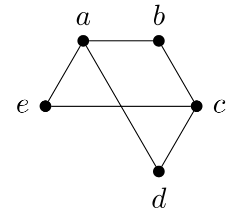
(d) Briefly justify each of (a)–(c) in Proposition 1.5.3 of EC1.
(e) Briefly justify each of (a)–(c) in Proposition 1.5.5 of EC1.

Exercise 36. Let A, B, C, and D be the graphs

$A =$ 

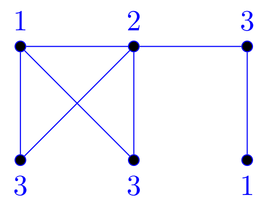
$B =$ 

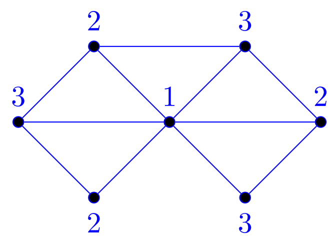
$C =$ 

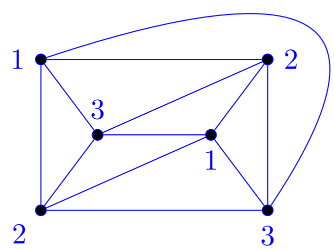
$D =$ 

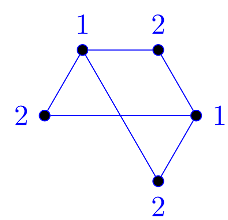
(a) Calculate the chromatic numbers for A, B, C, and D. For each, give an example of a vertex coloring of the corresponding graph using exactly χ colors.

Solution: Colorings indicated by 'colors' 1, 2, 3, . . . :

$A =$ 

$B =$ 

$C =$ 

$D =$ 

Since A, B, and B all have cliques of size 3, their chromatic numbers are at least 3, and D has edges, so its chromatic number is at least 2. Thus, by the colorings provided above, we have

$$\chi(A) = \chi(B) = \chi(C) = 3, \chi(D) = 2.$$

(b) What are the clique and independence numbers of A, B, C, and D? How do ω and $|V|/\alpha$ compare to χ for each graph?

Solution: The clique numbers are given by

$$\omega(A) = \omega(B) = \omega(C) = 3, \qquad \omega(D) = 2,$$

which exactly match the chromatic number. The independence numbers are given by

$$\alpha(A) = |\{c, d, e\}| = 3, \alpha(B) = |\{f, b, d\}| = 3, \alpha(C) = |\{a, f\}| = 2, \qquad \alpha(D) = |\{e, b, d\}| = 3,$$

so that

$$|V(A)|/\alpha(A) = 6/3 = 2, |V(B)|/\alpha(B) = 7/3 = 2.\bar{3}, |V(C)|/\alpha(C) = 6/2 = 3, |V(D)|/\alpha(D) = 5/3 = 1.\bar{6},$$

whose ceilings give sharp bounds for the chromatic numbers of B, C, and C, but not A.

(c) What are the chromatic numbers of

(i) $K_{m,n}$, (ii) C_n, (iii) W_n, and (iv) Q_n?

Solution:

$$\text{(i) } \chi(K_{m,n}) = 2, \qquad \text{(ii) } \chi(C_n) = \begin{cases} 2 & n \text{ even}, \\ 3 & n \text{ odd}, \end{cases}$$

$$\text{(iii) } \chi(W_n) = \begin{cases} 3 & n \text{ even}, \\ 4 & n \text{ odd}, \end{cases}, \qquad \text{and} \qquad \text{(iv) } \chi(Q_n) = 2.$$

(d) What are the clique and independence numbers of

(i) $K_{m,n}$, (ii) C_n, (iii) W_n, (iv) Q_n?

How do ω and $|V|/\alpha$ compare to χ for each graph? (You may need to break into cases.)

Solution:

$$\omega(K_{m,n}) = 2$$
$$\omega(C_n) = \begin{cases} 2 & n = 3, \\ 2 & n > 3 \end{cases}$$
$$\omega(W_n) = \begin{cases} 4 & n = 3 \\ 3 & n > 3 \end{cases}$$
$$\omega(Q_n) = 2.$$

$$\alpha(K_{m,n}) = \max(m, n)$$
$$\alpha(C_n) = \lfloor n/2 \rfloor$$
$$\alpha(W_n) = \lfloor n/2 \rfloor$$
$$\alpha(Q_n) = 2^{n-1}.$$

(e) What are necessary and sufficient conditions for a graph to have chromatic number (i) 1, and (ii) 2.

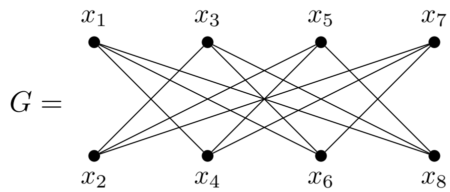
Solution: The chromatic number of G is 1 if and only if G has no edges. The chromatic number of G is 2 if and only if G is bipartite and contains an edge.

(f) For A, B, C, and D, pick an ordering of the vertices and color the graph using the greedy algorithm. Compare the colors you used to the chromatic number of the graph. Can you find an ordering that yields a coloring with too many colors?

(g) Explain why the clique number of the complement of a bipartite is no smaller than the number of vertices in each part. (Recall that the *parts* of a bipartite graph are the two collections of pairwise non-adjacent vertices.)

Solution: Since each part of a bipartite graph is an independent set, that set forms a clique in the complement.

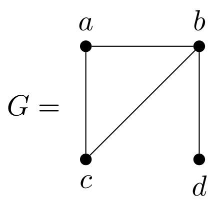
(h) Notice that the graph



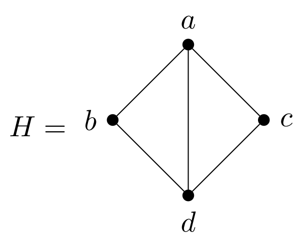
is bipartite, so should have chromatic number 2. Now color this graph using the greedy algorithm. How many colors did you need?

Answer: 4

Exercise 37. (a) Compute the number of ways to color the graph with each of 1, 2, ..., $|V|$ colors for



and



Solution:

	$\chi(\cdot, 0)$	$\chi(\cdot, 1)$	$\chi(\cdot, 2)$	$\chi(\cdot, 3)$	$\chi(\cdot, 4)$
G	0	0	0	$3 * 2^2$	$4 * 3^2 * 2$
H	0	0	0	$3 * 2$	$4 * 3 * 2^2$

(b) Calculate the chromatic polynomial for G above.

Answer: $\chi(G, t) = t(t-1)^2(t-2)$, $\chi(G, t) = t(t-1)(t-2)^2$.

(c) The chromatic polynomial for the cycle C_n is $\chi(C_n, k) = (k-1)^n + (-1)^n(k-1)$.

(i) Draw all the ways of coloring the 3-cycle with 3 colors. Then compute $\chi(C_3, 3)$ and compare your answers.

Solution: Order the vertices in clockwise order. Choose the color for the first vertex (3 choices), then the second (2 choices), and finally the third vertex (1 choice), giving 6 colorings. This matches

$$\chi(C_3, 3) = (3-1)^3 + (-1)^3(3-1) = 2^3 - 2 = 6.$$

(ii) How many ways are there to color the 5-cycle with 3 colors?

Solution:

$$\chi(C_5, 3) = (3-1)^5 + (-1)^5(3-1) = 2^5 - 2.$$

(iii) How many ways are there to color the 6-cycle with 2 colors?

Solution:

$$\chi(C_5, 3) = (2-1)^6 + (-1)^6(2-1) = 1 + 1 = 2.$$

(iv) Use $\chi(C_n, k)$ to verify that even cycles are bipartite and odd cycles are not.

Solution: For n even or odd, $\chi(C_n, 1) = (1-1)^n + (-1)^n(1-1) = 0$. For n even,

$$\chi(C_n, 2) = (2-1)^n + (-1)^n(2-1) = 1 + 1 = 2 \neq 0,$$

so $\chi(C_n) = 2$. For n odd,

$$\chi(C_n, 2) = (2-1)^n + (-1)^n(2-1) = 1 - 1 = 0,$$

so $\chi(C_n) > 2$.