Exercise 32. Complete the proof of the following theorem:
The following are equivalent for a simple graph G.

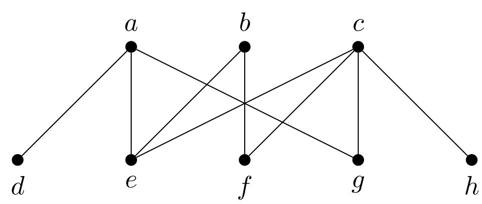
(1) G is a tree.
(2) G is a *minimal connected graph*, i.e. every edge in E is a cut edge.
(3) G is a maximally acyclic graph, i.e. G is acyclic, and adding any edge between two non-adjacent vertices creates a cycle.

Solution: (1) $\implies$ (2): Let $G = (V, E, \phi)$ be a tree, so that it is connected and acyclic. Let $e \in E$, and let $\phi(e) = \{u, v\}$. Suppose there is some path P from u to v that does not go through e. Then extending P by e results in a cycle, which is a contradiction. Thus e is a cut edge, and so G is minimally connected.
(2) $\implies$ (1): Let G be a minimally connected graph. Suppose G contains a cycle C, with e an edge in C. Then any walk which goes through e can be rerouted along $C - e$. Thus e is not a cut edge, a contradiction. Thus G is connected and acyclic, and therefore a tree.
(1) $\implies$ (3): Let $G = (V, E, \phi)$ be a tree, so that it is connected and acyclic. Consider an edge $e \in \bar{G}$, with $\phi(e) = \{u, v\}$. Since G is connected, there is a path P in G from u to v. Thus, in $G + e$, extending P by e results in a cycle. Thus G is maximally acyclic.
Not (1) $\implies$ not (3): Let $G = (V, E, \phi)$ be a graph that is not a tree. Then either G contains at least one cycle, implying G is not acyclic, or G is a forrest with at least two connected components. If G is a forrest, then let u and v be in different connected components. Thus $uv \notin E$, and there are no paths from u to v. Thus $G + e$ is also acyclic. In either case, G is not maximally acyclic.

Exercise 33. (a) How many spanning trees does C_5 have?

Solution: One corresponding to the deletion of each edge in C_5: 5.

(b) Let

$G =$ 

(i) Calculate diam(G) and rad(G).

Solution:

$d(u,v)$	a	b	c	d	e	f	g	h
a	0	2	2	1	1	3	1	3
b	2	0	2	3	1	1	3	3
c	2	2	0	3	1	1	1	1
d	1	3	3	0	2	4	2	4
e	1	1	1	2	0	2	2	2
f	3	1	1	4	2	0	2	2
g	1	3	1	2	2	2	0	2
h	3	3	1	4	2	2	2	0

$$\text{diam}(G) = d(d,h) = 4, \qquad \text{rad}(G) = 2 \text{ (see } e\text{)}$$

(ii) Fix the vertex a, and give $V_i = \{u \in V \mid d(u,a) = i\}$.

Solution:

$$\begin{aligned} V_0 &= \{a\} \\ V_1 &= \{d, e, g\} \\ V_2 &= \{b, c\} \\ V_3 &= \{f, h\}. \end{aligned}$$

(iii) Build a spanning tree using Method 1 from the notes using $v = a$ (show your steps!). What is the radius of the resulting tree?'

(iv) Find a central vertex v, i.e. one for which $\max_{u \in V} d(u, v) = \text{rad}(G)$, and build a spanning tree using Method 1 from the notes using that vertex (show your steps!). What is the radius of the resulting tree?

(v) Build a spanning tree using Method 2 from the notes starting with T_1 being the isolated vertex a (show your steps!).