Solutions for HW1

Exercise 1. Extended warmup.

(a) Write out the following sets explicitly.

$$[4]^2 = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$$

$$2^{[4]} = \{\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\\ \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}$$

$$S = \{(a,b) \mid a \in [2,4], b \in [-4,7]\}$$

$$= \left\{ \begin{array}{llllllllllll} (2,-4), & (2,-3), & (2,-2), & (2,-1), & (2,0), & (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), & (2,7), \\ (3,-4), & (3,-3), & (3,-2), & (3,-1), & (3,0), & (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), & (3,7), \\ (4,-4), & (4,-3), & (4,-2), & (4,-1), & (4,0), & (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), & (4,7) \end{array} \right\}$$

$[4]^2 \cap 2^{[4]} = \emptyset$

$[4]^2 \cap S = \{(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

$2^{[3]} \cup 2^{[4]} = 2^{[4]}$ (see above).

(b) For sets A and B, decide whether the following identities are **true or false**, and why.

(i) $A \cap B = B \cap A$: True. The statement "contained in A or B" is equivalent to "contained in B or A".

(ii) $A \cup B = B \cup A$: True. The statement "contained in A and B" is equivalent to "contained in B and A".

(iii) $A - B = B - A$: False. For example, let $A = \{x, y\}$ and $B = \{y, z\}$. Then $A - B = \{x\}$ and $B - A = \{z\}$.

(iv) $|A - B| = |A| - |B|$: False. For example, Let A and B be as above. Then $|A - B| = 1$ and $|A| - |B| = 2 - 2 = 0$.

(c) Answer the following counting problems, leaving your numerical answer unsimplified.

(i) A particular kind of shirt comes in two different cuts–male and female, each in three color choices and five sizes. How many different choices are made available?

Solution: Using product rule, we have $\boxed{2 * 3 * 5}$ choices.

(ii) On a 10-question true-or-false quiz, how many different ways can a student fill out the quiz if they answer all of the questions? if they might leave questions blank?

Solution: Using product rule, we have $\boxed{2^{10}}$ possibilities if every question is answered; and $\boxed{3^{10}}$ possibilities if they might leave some blank.

(iii) How many 3-letter words (not "real" words, just strings of letters) are there?

Solution: Using product rule, we have $\boxed{26^3}$ three-letter words.

(iv) How many 3-letter words are there that have no repeated characters?

Solution: Using product rule, we have $\boxed{26 * 25 * 24}$ words with no repeated characters.

(v) How many 3-letter words are there that have the property that if they start in a vowel then they don't end in a vowel?

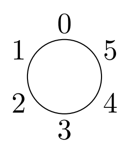
Solution: Discounting y, there are five vowels. There are $5^2 * 26$ (three-letter) words that start and end in a vowel, so using the complement rule, there are $\boxed{26^3 - 5^2 * 26}$ words that don't both start and end in a vowel.

Exercise 2 (EC 1.2)**.** Give as simple a solution as possible. Justify your answers (using words).

(a) How many subsets of the set $[10] = \{1, 2, \ldots, 10\}$ contain at least one odd integer?

Solution: There are 2^{10} subsets of $[10]$, and 2^5 subsets of $[10]$ that contain no odd integers (the number of subsets of $\{2, 4, 6, 8, 10\}$). So there are $\boxed{2^{10} - 2^5}$ subsets of $[10]$ contain at least one odd integer.

(b) In how many ways can six people be seated in a circle if two seatings are considered the same whenever each person has the same neighbors (not necessarily on the same side)? For example,



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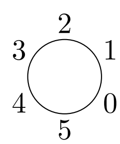

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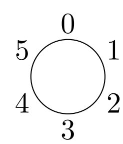
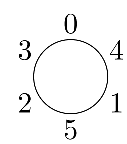


but not


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Solution: There are $6!$ permutations of $[6]$, but there are 2 orientations of each permutation around the table and 6 cyclic rotations of each oriented permutation. So there are $\boxed{6!/(2 * 6)}$ non-equivalent seatings.

(c) A *permutation* of a finite set S is a bijective map $w : S \to [n]$, where $n = |S|$.

(i) How many permutations $w : [6] \to [6]$ are there? $\boxed{6!}$

(ii) How many permutations $w : [6] \to [6]$ satisfy $w(1) \neq 2$?

Solution: There are 5 choices for $w(1)$ (any of $[6] - \{2\}$), 5 remaining choices for $w(2)$ (any of $[6] - \{w(1)\}$), 4 choices for $w(3)$, $\ldots$, one choice for $w(6)$. So there are $5 * 5!$ such permutations.
Alternatively, there are $5!$ permutations with $w(1) = 2$, so there are $6! - 5!$ such permutations.
Check: $6! - 5! = 6 * 5! - 5! = 5 * 5! \checkmark$.

A *cycle* of a permutation is a sequence $(c_1, c_2, \ldots, c_\ell)$ such that

$$w : c_1 \mapsto c_2, \quad w : c_2 \mapsto c_3, \quad \ldots \quad w : c_{\ell-1} \mapsto c_\ell, \quad w : c_\ell \mapsto c_1.$$

For example, the permutation $w : [4] \to [4]$ given by

$$1 \mapsto 4, \quad 2 \mapsto 2, \quad 3 \mapsto 1, \quad 4 \mapsto 3$$

has exactly two cycles: $(1, 4, 3)$ and (2).

(iii) How many permutations $w : [6] \to [6]$ have exactly one cycle? [Hint: question (b) $\times 2$.)

Solution: Start at 1. There are five values c_2 to which 1 can map, then four values c_3 to which c_2 can map, and so on. Whatever c_6 is, it maps to 1. So there are $\boxed{5!}$ permutations that have exactly one cycle. This is like question (b), except that $1 \mapsto c_2 \mapsto c_3 \mapsto c_4 \mapsto c_5 \mapsto c_6 \mapsto 1$ is different from the cycle $1 \mapsto c_6 \mapsto c_5 \mapsto c_4 \mapsto c_3 \mapsto c_2 \mapsto 1$. So there are twice as many of such cycles as non-equivalent seating arrangements, i.e. $6!/6 = 5! \checkmark$.

(iv) How many permutations $w : [6] \to [6]$ have exactly two cycles of length 3?

Solution: First, partition $[6]$ into two subsets of size 3, of which there are $\frac{1}{2}\binom{6}{3}$ ways. Then for the set containing 1, arrange in a cycle, of which there are $2!$ ways. Similarly, there are $2!$ ways to arrange the set not containing 1 into a cycle. So there are $\boxed{\frac{1}{2}\binom{6}{3}(2!)^2}$ such cycles.

(d) There are four people who want to sit down, and six distinct chairs in which to do so. In how many ways can this be done?

Solution: First, choose the 4 seats that will be taken, of which there are $\binom{6}{4}$ ways. Then there are $4!$ ways to arrange the people amongst those seats. So there are $\boxed{\binom{6}{4}4!}$ seating arrangements in total.

Exercise 3. (a) Explain why $\binom{n}{k} = \frac{(n)_k}{k!}$ directly using product and division rules.

Solution: Consider size k subsets of $[n]$. Choose the k element subset one element at a time, for which product rule tells us there are $n * (n-1) \cdots (n-k+1) = (n)_k$ ways. However, there are $k!$ rearrangements of this process. So division rule tells us that there are $(n)_k/k!$ size-k subsets of $[n]$.

(b) Give a combinatorial proof of the identity

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n},$$

where $a, b, n \in \mathbb{N}$ and $a, b \geq n$. [Hint: Consider two disjoint sets A and B, with $|A| = a$ and $|B| = b$. How many subsets does $A \sqcup B$ have?]

Solution: Let A and B be disjoint sets with $|A| = a$ and $|B| = b$. Then $|A \sqcup B| = a + b$. Therefore, by definition, the number of size-n subsets of $A \sqcup B$ is $\binom{a+b}{n}$. On the other hand, we can build all subsets S of $A \sqcup B$ by choosing a subset S_A of A followed by a subset S_B of B and then taking their union $S = S_A \sqcup S_B$. If $|S_A| = i$ and $|S| = n$, then since A and B are disjoint, we have $|S_B| = n - i$. So in this way, the number of subsets of $A \sqcup B$ is $\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}$. Since both values express the number of subsets of $A \sqcup B$ of size n, we have

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}.$$