Exam #2 Extra Credit

Math 365 – Daugherty

Due in class May 1, 2019

Name: ____

Instructions:

- Your answers **must be typed**, with the exception that you may fill in pictures or equations by hand as necessary. You are highly encouraged to use LaTeX, but you may use other word-processing programs like Word if you prefer. Either way, please follow these three formatting requirements:
  i. include this page as a cover page of your document;
  ii. include the question prompts before your answers;
  (Tip: even if you're not using LaTeX, you can download the .tex file to copy and paste the wording.)
  iii. start each of the four problems on a new page. (For example, you can use your own discretion about whether or not part (b) of problem 1 needs to be started on a new page, but problem 2 should definitely start at the top of a new page; generally avoid splitting a proof across two pages if possible).
- Your answers must be written in **complete sentences**. The point is to explain your understanding, not to complete computations alone. Revisit the handout "Guidelines for Good Mathematical Writing" if necessary. Your grade will depend on the *quality* of your exposition, not just its validity.
- Your **audience** for this assignment is a math major who has not taken 365. In other words, you may assume general mathematical fluency of your reader, but not that they already know what you're talking about. Think about explaining what 365 is about to a classmate in a different course.
- You may talk to your professor or classmates when working some of these out, but **your answers must be your own**. In particular, it is strongly suggested that you do not collaborate on your examples. If you take an example out of the book or any other resource, **cite your source**.
- Your **total grade** on Exam 2 will be an average of your score on this assignment and your original Exam 2 grade, with the following exceptions—your total grade will not exceed 80%, nor will it decrease as a result of a low grade on this assignment.

Problem #	Out of	Score
1	20	
2	50	
3	10	
4	20	
total	100	

1. *Counting.*

(a) Give the combinatorial meaning of each of the following functions, and give an example of a word problem where it would arise.

(i) $\binom{n}{k}$

(ii) $S(n, k)$

(iii) $P(n, k)$

(iv) $p_k(n)$

(b) Summarize the content of Sections 6.1, 3, and 5. In your summary, be sure to explain the big picture, give the major kinds of problems in general terms, and include illustrative examples. You may refer to part (a).

2. *Combinatorial proofs.* (For example, see Section 6.4, proofs of Theorems 2, 3, and 4.)

(a) Explain the general format of a combinatorial proof.

(b) Adapt the proof of Theorem 3 in Section 6.4 to prove the identity in Corollary 4,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

[Namely, they prove an identity for general m, n, and r in Theorem 3 and then say if you set $m = n = r$, you get Corollary 4 for free. I want you to rewrite the proof of Theorem 3, adapting it to just prove Corollary 4 instead.]

(c) Give a combinatorial proof of the following identity:

$$n2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k.$$

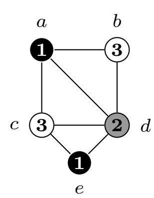
[Hint: Think about picking a club and its president.]

(d) A *graph coloring* (with k colors) is a way to color the vertices of a graph so that adjacent vertices get colored differently. For example, if I have three colors,

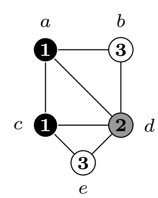
black 1 , gray 2 , and white 3 ,

then

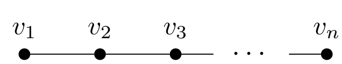
this is a valid coloring:



but this is *not* a valid coloring:



since a and c are adjacent and both colored black. Let c_n be the number of ways to color the n-vertex path

$P_n =$ 

with k colors. Give a combinatorial proof of the recursive identity

$$c_n = (k-1)c_{n-1}.$$

(e) Let $\sigma_m(n)$ be the number of surjective functions from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. Assuming $m \geq n > 1$, give a combinatorial proof of the recursive identity

$$\sigma_m(n) = n^m - \sum_{k=1}^{n-1} \binom{n}{k} \sigma_m(k).$$

(f) Explain the big-picture idea of what Section 8.1 is about.

3. Summarize the content of Section 8.2, including both a big-picture explanation and a brief guide to solving recurrences of various types. Include examples.

4. *Generating functions.*

(a) Summarize the content of Section 8.4, including a big-picture explanation (what's a generating function and why do we care) and examples of the various ways we encounter generating functions.

(b) Explain why each of the following generating functions model the corresponding counting problem (i.e. justify the answers to the last problem on the exam).

(i) The number of integer partitions of n has the generating function

$$\prod_{k=1}^{\infty} \frac{1}{1 - x^k}.$$

(ii) The number of integer partitions of n with at most one part of each length has the generating function

$$\prod_{k=1}^{\infty} (1 + x^k).$$

(iii) The number of ways to make change for \$$n$ with \$1's, \$5's, and \$10's has the generating function

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{10}}\right).$$

(iv) The number of non-negative integer solutions to the equation $x_1 + x_2 + x_3 = n$, where $x_1, x_2, x_3 \leq 5$, has the generating function

$$\left(1 + x + x^2 + x^3 + x^4 + x^5\right)^3.$$