Answers (not solutions) for Ch6 Supp. Exercises (pp 441-443)

minimal, if any, justification

3) 3^{100}

4) $2^7 + 2^6 - 2^3$

5) $\binom{10}{3}2^7 + \binom{10}{4}2^6 - \binom{10}{3}\binom{10-3}{4}1^{10-4-3}$

6) $9 \cdot 10^4$

7) (a) $\binom{3+28-1}{3}$

(b) $28 \cdot 8 \cdot 12$

(c) $\binom{3+28-1}{3}\binom{8}{2}\binom{12}{3}$

8) (a) $9 \cdot 10^2$

(b) $9 \cdot 10^2 + 9$

(c) $\underbrace{1000}_{\text{tot}} - (\underbrace{1}_{1000} + \underbrace{9^3}_{\text{\_ \_ \_ fill w/ 0-8}} - \underbrace{1}_{000})$

(d) $5^3 - \underbrace{1}_{000}$

(e) $\underbrace{9}_{55\_,\ \_\neq 5} + \underbrace{9}_{\_55,\ \_\neq 5} + \underbrace{1}_{555}$

(f) $\underbrace{9}_{\underline{x}} + \underbrace{9 \cdot 10}_{\underline{x}\,\underline{y}\,\underline{x}}$, $x \neq 0$

9) (a) $\underset{\text{1 dig}}{0} + \underset{\substack{\text{2 dig}\\ -0}}{9} + \underset{--0}{9\cdot 10} + \underset{-0-}{9\cdot 10} + \underset{100}{3}$

(b) $\underset{1}{1} + \underset{\substack{-1\\ \neq 0}}{9} + \underset{1-}{10} + \underset{--1}{9\cdot 10} + \underset{-1-}{9\cdot 10} + \underset{1--}{10^2} + \underset{1000}{1}$

(c) & (d) $1+9+10+9\cdot 10+9\cdot 10+10^2$

10) Pidgeonhole: $\lceil N/12 \rceil \geq 6 \iff N/12 > 5$, $\boxed{N = 5\cdot 12+1}$

11) $3\cdot 213$

12) $7\cdot 12+1$

13) 10 positive integers ≤ 50 : $A = \{a_1, a_2, \dots, a_{10} \mid 0 < a_i \leq 50\}$
\# 5-element subsets: $\binom{10}{5} = 252$
Possible sums of 5 distinct elements from $\{1, \dots, 50\}$:

min:
$1+2+3+4+5 = 15$

max
$46+47+48+49+50 = 240$

total possibilities: $240-15+1 = 226$

Since \#{5-elt subsets of A} > \#{sums of 5-elt subsets of {1,...,50}},

At least two 5-elt subsets of A must have the same sum.

14) $\left\lceil \frac{20\cdot k}{550} \right\rceil \geq 2 \iff \frac{20\cdot k}{550} > 1 \iff k > \frac{55}{2}$, so $\boxed{k = 28}$

15) (a) $(52-4)+2$

(b) $\max(52-4+2,\ 4+1) = 52-4+2$

(c) $13+1$

(d) $13+3+1$

20) 100^5

21) (a) $\binom{20}{12}$

(b) $\binom{20}{1} = 20$

(c) $\binom{12+20-1}{12}$

(d) $\binom{12+20-1}{12} - 20$

(e) $\binom{12-6+20-1}{12-6}$

(f) $\sum_{K=0}^{6} \binom{(12-K)+(20-1)-1}{12-K}$

K = # blueberry-filled, $12-K$ = # remaining choices

$20-1$ = # varieties not bb-filled

22) (a) $110 = P(n,2) = n(n-1) \Rightarrow n = 11$

(solve $n^2 - n - 110 = 0$: $n = 11, -10$)

not pos int

(b) $5040 = P(n,n) = n! \Rightarrow n = 7$

(c) $n(n-1)(n-2)(n-3) = P(n,4)$
$= P(n,2) \cdot 12$
$= 12(n)(n-1)$

so $(n-2)(n-3) = 12$
$n^2 - 5n + \underbrace{6 - 12}_{-6} = 0$
$(n-6)(n+1) =$ so $\boxed{n=6}$
($n=-1$ not pos)

23) (a) $45 = C(n,2) = \frac{1}{2}n(n-1) \Rightarrow \boxed{n=10}$

(b) $\frac{1}{6}n(n-1)(n-2) = C(n,3)$
$= P(n,2) = n(n-1)$

so $\frac{1}{6}(n-2) = 1$
$n-2 = 6$
$\boxed{n=8}$

(c) $n(n-1)(n-2)(n-3)(n-4) \cdot \frac{1}{5!}$
$= C(n,5) = C(n,2) = \frac{1}{2}n(n-1)$

Stop! This only happens in $\binom{n}{k} = \binom{n}{n-k}$ case.

$5 = n-2 \Rightarrow \boxed{n=7}$

24) Show $n, r \in \mathbb{Z}_{\geq 0}$, $n \geq r$, then

$$P(n+1, r) = \frac{P(n,r)(n+1)}{n+1-r}$$

<u>Combinatorial Proof outline:</u>

LHS: # r-permutations of $n+1$ things

RHS: Build an r-perm of $\{1, \ldots, n+1\}$ as follows:

① Pick 1^{st} elt : $n+1$ ways
(ex: 3)

② Add on r-perm of remaining $(n+1)-1=n$ elts: $P(n,r)$ ways
(ex: $\boxed{3}\,2145$)

③ Forget $(r+1)^{th}$ elt of the resulting $(r+1)$-perm of $n+1$ things
(ex: $3214\cancel{5}$)

Since there are, for any fixed r-perm of $n+1$, a total of $n+1-r$ things that forgotten last elt could have been, there are

$$\frac{(n+1)P(n,r)}{n+1-r}$$ r-perms of $n+1$.

<u>Algebraic Proof</u>

$$\frac{P(n,r)(n+1)}{n+1-r} = \frac{(n+1)\overbrace{(n)(n-1)\cdots(n+1-r)}^{P(n,r)}}{n+1-r}$$

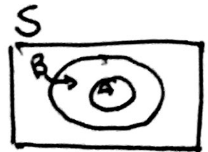
$$= (n+1)n(n-1)\cdots(n+1+1-r)$$

$$= P(n+1, r).$$

$|S| = n$

25) Count $\# \{ (A, B) \mid A \subseteq B \subseteq S \}$.

pf If $A \subseteq B \subseteq S$,



then S is partitioned into 3 disjoint subsets:

$$A \;,\; B - A \;,\; S - B.$$

(since $A \cup (B - A) = B$, and
$B \cup (S - B) = S$, so
$A \cup (B-A) \cup (S-B) = S$
and $A \cap (B-A) = \emptyset$,
$A \cap (S-B) = \emptyset$, and
$B \cap (S-B) = \emptyset$.)

So choosing subsets $A \subseteq B \subseteq S$ is the same as choosing which of A, $B - A$, and $S - B$ to place each $x \in S$ into.

result: $\boxed{3^n}$.

27) $n, r \in \mathbb{Z}$, $1 \le r < n$.

Show
$$C(n, r-1) = C(n+2, r+1) - 2C(n+1, r+1) + C(n, r+1)$$

combinatorial proof outline

LHS: #{r−1 subsets of size-n set}

RHS: let S be a set of size n,
let $x, y \notin S$, so that $|S \cup \{x,y\}| = n+2$.

So a size $r-1$ subset of S is the same as a size $r+1$ subset of $S \cup \{x,y\}$ that contains x & y.

There are

(total # r+1 subsets of $S \cup \{x,y\}$) − (subsets not containing x, y, or both)

(total # r+1 subsets of $S \cup \{x,y\}$) $= C(n+2, r+1)$

(subsets not containing x, y, or both) = (# subsets not containing {x}) + (# subsets not containing {y}) − (# subsets not containing {x, y})

$$= \boxed{C(n+1, r+1) + C(n+1, r+1) - C(n, r+1)}$$

Total:
$$C(n+2, r+1) - 2C(n+1, r+1) + C(n, r+1).$$

28 Use induction to show

$$\sum_{j=2}^{n} C(j,2) = C(n+1,3) \quad (*)$$

pf Let $P(n)$ be $(*)$.

Base case: $P(2)$ says

$$\sum_{j=2}^{2} C(j,2) = C(2,2) = 1 = C(2+1,3). \checkmark$$

Goal Assume $P(n)$ for fixed $n \geq 2$, and show $P(n+1)$ which is

$$\sum_{j=2}^{n+1} C(j,2) = C(n+2,3).$$

Inductive step: Assume $P(n)$ for fixed $n \geq 2$.

Then

$$\begin{aligned}\sum_{j=2}^{n+1} C(j,2) &= \left(\sum_{j=2}^{n} C(j,2)\right) + C(n+1,2) \\ &\overset{IH}{=} C(n+1,3) + C(n+1,2) \\ &= C(n+2,3) \text{ by Pascal's identity.}\end{aligned}$$

Conclusion Since $P(2)$ holds and $P(n)$ implies $P(n+1)$ for $n \geq 2$, we have $P(k)$ holds for $k = 2, 3, \ldots$ $(k \in \mathbb{Z}_{\geq 2})$.

29) Show

$$\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n \qquad \text{for } n \in \mathbb{Z}.$$

Note: for $n < 0$, this does not hold!
Prove for $n \geq 0$.

Pf Binomial thm says

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Plug in $x=3$, $y=1$ to get

$$4^n = (3+1)^n = \sum_{k=0}^{n} \binom{n}{k} 3^k 1^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} 3^k. \qquad \blacksquare$$

(For a combinatorial pf, count # ways to answer a multiple choice exam w/ 3 choices per question.
LHS: Let k = # left blank
RHS: just count)

33. If exactly 2 01's, that means
(maybe some 1's) followed by
(some 0's) followed by
(some 1's) — " —
(some 0's) — " —
(some 1's).

Cases:

starts w/ 1
(ie $11\ldots10\ldots\underbrace{01}_{1^{st}01}\ldots10\ldots\underbrace{01}_{2^{nd}01}\ldots1$)

n stars
4 bars
at least one star per region:
$$\binom{(n-5)+4}{4}$$

(for example,
$***|*|**|***|*$
means
$111\underbrace{01}_{1^{st}01}1000\underbrace{01}_{2^{nd}01}$)

starts w/ 0
(ie $0\ldots\underbrace{01}_{1^{st}01}\ldots10\ldots\underbrace{01}_{2^{nd}01}\ldots1$)

n stars
3 bars
at least one star per region:
$$\binom{(n-4)+3}{3}$$

(for example,
$**|****|*|***$
means
$0\underbrace{01}_{1^{st}01}111\underbrace{01}_{2^{nd}01}11$)

Total: $\binom{n-1}{4}+\binom{n-1}{3}=\binom{n}{4}$ Ah hah!

Alternatively: ~~[illegible]~~
Start w/ 1. Of the n bits, pick 4 places to switch bit type (ok to switch @ beginning, but not @ end; don't switch more than once per bit). : $\binom{n}{4}$

ex: — — — — — — — — — — means $00\underbrace{01}000\underbrace{01}11$

switch

35) $\binom{20}{8}\binom{20-8}{3}\binom{20-8-3}{4}\binom{20-8-3-4}{5}$

$$= \frac{20!}{8!\,3!\,4!\,5!}$$

36) $\frac{6!}{6} = 5!$

37) 5^{24}

38) $\binom{(12-3\cdot 3)+(3-1)}{3-1}$

39) (a) $\binom{(17-(1+2+3))+(3-1)}{(3-1)}$

(b) $\sum_{x_1=0}^{5} \binom{((17-x_1)-5)+(2-1)}{2-1} = \sum_{x_1=0}^{5} (13-x_1)$

$= 6\cdot 13 - \sum_{x_1=1}^{5} x_1$.

(fix x_1, count solns to $x_2+x_3 = 17-x_1$ w/ $x_3>5$)

(c) Count tot $x_3>5$ $-$ $\#\{x_1 \geq 4, x_2 \geq 3\}$ $x_3>5$

$\binom{17-5+(3-1)}{3-1} - \binom{17-(5+3+2)+(3-1)}{3-1}$

40) (a) $\#P=3,\ \#E=2,\ \#R=2,\ \#C=1,\ \#O=1,\ \#N=1$

$$\frac{(3+2+2+1+1+1)!}{3!\,2!\,2!\,1!\,1!\,1!}$$

(b) $$\frac{(1+2+2+1+1+1)!}{1!\,2!\,2!\,1!\,1!\,1!}$$

(c) $$\frac{(1+2+2+1+1+1)!}{1!\,2!\,2!\,1!\,1!\,1!}$$ (think of 'PPP' as one letter)

41) (a) $$\sum_{k=0}^{4}\binom{10}{k}$$

(b) $$\sum_{k=8}^{10}\binom{10}{k}$$

(c) $$\sum_{k\in\{1,3,5,7,9\}}\binom{10}{k}$$

42) $$26\cdot\Big[\underbrace{3}_{\{1,1,2\}\text{ permuted}}+\underbrace{3}_{\{1,2,2\}\text{ permuted}}+\underbrace{3!\cdot 8}_{\{1,2,x\},\ x\neq 1,2\text{ permuted}}\Big]$$

43) [redacted] $$\binom{n-m+m-1}{m-1}=\binom{n-1}{m-1}$$

44) Think: 8 Girls = stars
6 Boys = bars

w/ [redacted] at least one star between each pair of bars

$$\binom{8-(6-1)+6}{6}=\binom{9}{6}$$

then pick labels of |'s w/ specific boys & *'s w/ specific girls

total: $\boxed{\binom{9}{6}8!\,6!}$