Math 365 – WORKSHEET – Wednesday 3/27/19
See the last few pages for some review!

1. Write the generating functions for the following sequences, in both their series form and closed form (the simplified form). Assume in each case that the sequence starts at a_0. For example, the sequence $1, 2, 3, 4, 4, 4, 4, \ldots$ has the generating function

$$\underbrace{1 + 2x + 3x^2 + \sum_{n=3}^{\infty} 4x^n}_{\text{series form}} = 1 + 2x + 3x^2 + 4x^3 \sum_{n=0}^{\infty} x^n = \underbrace{1 + 2x + 3x^2 + \frac{4x^3}{1-x}}_{\text{closed form}}.$$

Start by writing the *sequence itself* in a closed form. For example, the above sequence is $a_0 = 1, a_1 = 2, a_2 = 3,$ and $a_n = 4$ for $n \geq 3$.

(a) $5, 5, 5, 5, 5, \ldots$

(b) $1, 3, 9, 27, \ldots$

(c) $1, -1, 1, -1, 1, -1, \ldots$

(d) $1, -2, 4, -8, 16, \ldots$

(e) $1, 0, 0, 1, 1, 1, 1, 1, \ldots$

(f) $0, 0, 0, 2, 2, 2, 2, 2, \ldots$

(g) $1, 3, -2, 5, 10, 20, 40, 80, \ldots$

(h) $1, 2, 3, 4, 5, \ldots$

(i) $3, 6, 9, 12, 15, \ldots$

(j) $2, 6, 12, 20, 30, 42, \ldots$ (hint: $6 = 3 * 2, 12 = 4 * 3, 20 = 5 * 4, \ldots$)

(k) $1, 0, -2, 2, 6, 12, 20, 30, 42, \ldots$

2. For each of the following, write the generating function for which the coefficient of x^r is the answer.
   (a) How many ways can postage of r cents can be pasted on an envelope using 3-cent, 4-cent, and 20-cent stamps, if the arrangement of the postage doesn't matter?

   (b) How many ways can you make change for $100 using $5, $10, $20, and $50 bills?

   (c) How many ways can you make change for $100 using $5, $10, $20, and $50 bills, if you only have 4 of each kind of bill?

   (d) How many ways can you make change for $100 using $5, $10, $20, and $50 bills, if you have to use at least one of each kind of bill, but you only have one $50?

3. Suppose you have 5 pennies, 3 nickels, and a dime in your pocket. When reaching in to your pocket to pull out change, you do so totally randomly (any subset of coins is equally likely).
   (a) What is the generating function for the total value of any set of change you might pull out?

   (b) What value(s) of change are you most likely to pull out?
   [Hint: use something like wolframalpha.com to expand your polynomial]

4. For each of the following recursion relations, suppose that $G(x)$ is the generating function for the corresponding solution. Write an expression in terms of $G(x)$ and other familiar generating functions that will allow you to solve for $G(x)$. For example, if

$$a_n = 8a_{n-1} + 10^{n-1}, \qquad \text{then} \qquad \boxed{G(x) = a_0 + 8xG(x) + \frac{x}{1-10x}.}$$

The relevant calculation here is

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \qquad = a_0 + \sum_{n=1}^{\infty} a_n x^n \quad = a_0 + \sum_{n=1}^{\infty} (8a_{n-1} + 10^{n-1})x^n$$

$$= a_0 + 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \quad = a_0 + 8\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 10^n x^{n+1}$$

$$= a_0 + 8xG(x) + x\sum_{n=0}^{\infty} 10^n x^n \quad = a_0 + 8xG(x) + x\frac{1}{1-10x}.$$

(See p. 7 for another example)

(a) $a_n = 7a_{n-1}$

(b) $a_n = a_{n-1} + a_{n-2}$

(c) $a_n = a_{n-1} + 2a_{n-2}$

(d) $a_n = a_{n-1} + 2a_{n-2} + 2^n$

(e) $a_n = a_{n-1} + 2a_{n-2} + n$

(f) $a_n = a_{n-1} + 2a_{n-2} + 2^n + n + 7.$

5. Use your answer to part (b) of the previous problem to solve for a_n when $a_n = a_{n-1} + a_{n-2}$, $a_0 = 0$ and $a_1 = 1$. Compare this the solution we found in Section 8.2 (example 4 in the book).

Generating functions quick facts:

The most important series are

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \qquad \frac{1-x^n}{1-x} = \sum_{k=0}^{n-1} x^k,$$

$$e^x = \sum_{k=0}^{\infty} x^k/k!, \qquad \text{and} \qquad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

Building new functions from old: If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$, we have

$$f(x)+g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k, \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k.$$

$$f(cx) = \sum_{k=0}^{\infty} a_k c^k x^k, \qquad f(x^c) = \sum_{k=0}^{\infty} a_k x^{ck} = \sum_{n=0}^{\infty} A_n x^n, \text{ where } A_n = \begin{cases} a_{n/c} & \text{if } n \text{ is a mult. of } c, \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{d}{dx} f(x) = \sum_{k=0}^{\infty} a_k k x^{k-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n,$$

$$\int f(x)dx = \left(\sum_{k}^{\infty} a_k \frac{x^{k+1}}{k+1} \right) + c = c + \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n,$$

where c is any real number.

Solving for the unknown c when integrating: For example, since $\int -\frac{1}{1-x}\, dx = \ln|1-x| + C$, the series for $\ln|1-x|$ is

$$\ln|1-x| = c + \sum_{n=1}^{\infty} -\frac{1}{n} x^n$$

for some c. How do you figure out what exactly c is? Well, we know that

$$\ln|1-x| \Big|_{x=0} = \ln|1| = 0.$$

So, on the right hand side, we must also have that

$$0 = \left(c + \sum_{n=1}^{\infty} -\frac{1}{n} x^n \right) \Big|_{x=0} = c + 0 = c.$$

So

$$\ln|1-x| = \sum_{n=1}^{\infty} -\frac{1}{n} x^n.$$

Using generating functions to solve recurrence relations: We always start by defining

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=1}^{\infty} a_n x^n.$$

Example: if $a_n = 3a_{n-1} + 2^n$, then

$$\begin{aligned}
G(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= a_0 + (3a_0 + 2)x + (3a_1 + 2^2)x^2 + (3a_1 + 2^3)x^3 + \cdots \\
&= a_0 + (3a_0 x + 3a_1 x^2 + 3a_2 x^3 + \cdots) + (2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots) \\
&= a_0 + 3x(\underbrace{a_0 + a_1 x + a_2 x^2 + \cdots}_{G(x)}) + 2x(\underbrace{1 + 2x + 2^2x^2 + 2^3x^3 + \cdots}_{=\sum_{k=1}^{\infty}(2x)^k = \frac{1}{1-y}\big|_{y=2x}}) \\
&= a_0 + 3xG(x) + 2x\frac{1}{1-2x}.
\end{aligned}$$

So

$$(1-3x)G(x) = a_0 + \frac{2x}{1-2x} = \frac{a_0 + 2(1-a_0)x}{1-2x}.$$

Now, finally solve for a_n as follows:

$$\begin{aligned}
G(x) &= \frac{a_0 + 2(1-a_0)x}{(1-2x)(1-3x)} = -2\frac{1}{1-2x} + (a_0+2)\frac{1}{1-3x} \quad \text{(\underline{using partial fractions} - see below!)} \\
&= -2\sum_{n=0}^{\infty} 2^n x^n + (a_0+2)\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty}((-2)2^n + (a_0+2)3^n)x^n.
\end{aligned}$$

Thus, $\boxed{a_n = (-2)2^n + (a_0+2)3^n}$.

The partial fractions step: We know that there must be some numbers A and B such that

$$\frac{a_0 + 2(1-a_0)}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x}.$$

Cross-multiplying, we get

$$a_0 + 2(1-a_0) = A(1-3x) + B(1-2x) = (A+B) - (3A+2B)x.$$

So

$$a_0 = A + B, \qquad \text{meaning that } B = a_0 - A.$$

And

$$2(1-a_0) = -3A - 2B = -3A - 2(a_0 - A) = -A - 2a_0.$$

So $A = -2$ and $B = a_0 - (-2) = a_0 + 2$.

Examples of finite generating functions used in counting problems:
Example: Use a generating function to answer the question "How many non-negative integer solutions are there to

$$e_1 + e_2 + e_3 = 10$$

where e_2 is a multiple of 2 and e_3 is a multiple of 3?"

The answer is the same as the coefficient of x^{10} in

$$\underbrace{(1 + x + x^2 + \cdots)}_{\text{choices of possible } x^{e_1}} \underbrace{(1 + x^2 + x^4 + x^6 + \cdots)}_{\text{choices of possible } x^{e_2}} \underbrace{(1 + x^3 + x^6 + x^9 + \cdots)}_{\text{choices of possible } x^{e_3}},$$

which is the same as the coefficient of x^{10} in

$$\underbrace{(1 + x + x^2 + \cdots + x^{10})}_{\text{choices of possible } x^{e_1}} \underbrace{(1 + x^2 + x^4 + \cdots + x^{10})}_{\text{choices of possible } x^{e_2}} \underbrace{(1 + x^3 + x^6 + x^9)}_{\text{choices of possible } x^{e_3}},$$

since we would never use any terms that came from x^a for $a > 10$.

Example: How many integer partitions are there of 5?

This is the same as the coefficient of x^5 in

$$\begin{aligned}
&(1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^2 + x^4)(1 + x^3)(1 + x^4)(1 + x^5) \\
&\quad = \big((x^1)^0 + (x^1)^1 + (x^1)^2 + (x^1)^3 + (x^1)^4 + (x^1)^5\big) \quad \text{(possible choices of } x \text{ to \# of pts of length 1)} \\
&\qquad \big((x^2)^0 + (x^2)^1 + (x^2)^2\big) \qquad \text{(possible choices of } x^2 \text{ to \# of pts of length 2)} \\
&\qquad \big((x^3)^0 + (x^3)^1\big) \qquad \text{(possible choices of } x^3 \text{ to \# of pts of length 3)} \\
&\qquad \big((x^4)^0 + (x^4)^1\big) \qquad \text{(possible choices of } x^4 \text{ to \# of pts of length 4)} \\
&\qquad \big((x^5)^0 + (x^5)^1\big) \qquad \text{(possible choices of } x^5 \text{ to \# of pts of length 5)}
\end{aligned}$$