Math 365 – Wednesday 5/1/19

Exercise 57. (a) The chromatic polynomial for the cycle C_n is $\chi(C_n, t) = (t-1)^n + (-1)^n(t-1)$.

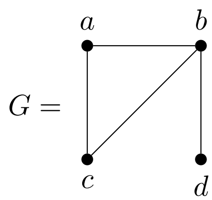
(i) Draw all the ways of coloring the 3-cycle with 3 colors. Then compute $\chi(C_3, 3)$ and compare your answers.

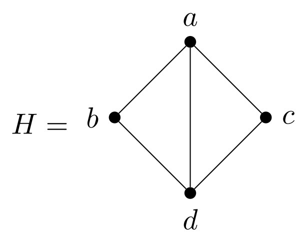
(ii) How many ways are there to color the 5-cycle with 3 colors?

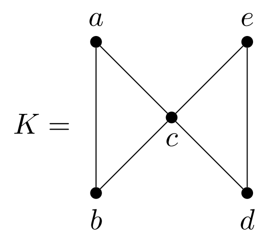
(iii) How many ways are there to color the 6-cycle with 2 colors?

(iv) Use $\chi(C_n, t)$ to verify that even cycles are bipartite and odd cycles are not.

(b) For G and H below, compute the number of ways to color the graph with a palate of 1 color, of 2 colors, of 3 colors, and of 4 colors. For K, compute the number of ways to color the graph with a palate of 1 color, of 2 colors, of 3 colors, of 4 colors, and of 5 colors.







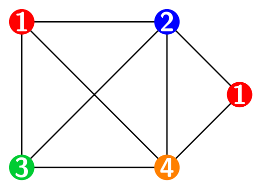
[Hint: For all three, break into cases based on how many colors you actually use. Then for any coloring using exactly k colors, there will be $k!$ colorings with that "color pattern" – so instead of listing all colorings, list color patterns and then account for how many colorings correspond to each pattern. For K, you might want to use some of what you've already computed about colorings of C_3 to count colorings of K efficiently.]

(c) Calculate the chromatic polynomial for G and for H.

(d) Explain why $\chi(K_n, t) = t(t-1)(t-2)\cdots(t-(n-1))$.

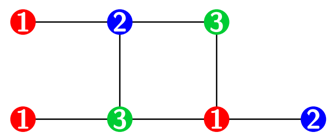
Last time:

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

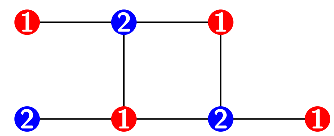


using 4 colors

$\chi = 4$



using 3 colors



using 2 colors

$\chi = 2$

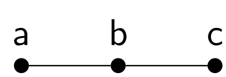
The chromatic number of a graph G, denoted $\chi(G)$, is the least number of colors needed for a coloring of this graph.

To calculate, argue that the graph can't be colored in $\chi - 1$ colors, and then give a coloring with exactly χ colors.

The chromatic polynomial

Question: Given a palate of t colors, how many ways are there to color a (labeled) graph using that palate? (You don't have to use all the colors at once.)
Let $\chi(G, t)$ be the number of colorings with a palate of t colors.

For example, the path graph

$P_3 =$ 

cannot be colored at all with 0 or 1 colors. So

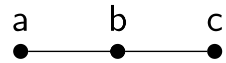
$$\chi(P_3, 0) = \chi(P_3, 1) = 0.$$

With 2 colors (say red and blue), it can be colored in 2 ways:

So $\chi(P_3, 2) = 2$.

The chromatic polynomial

Let $\chi(G, t)$ be the number of colorings with a palate of t colors.

For example, the path graph $P_3 =$  has

$\chi(P_3, 0) = \chi(P_3, 1) = 0$, and $\chi(P_3, 2) = 2$.
With 3 colors, we break into cases: use all 3 colors, or only use 2.
Using exactly 2 colors, this can happen in $\chi(P_3, 2) * \binom{3}{2} = 6$ ways:

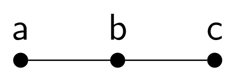
color w/ red (1) and blue (2):

color w/ red (1) and green (3):

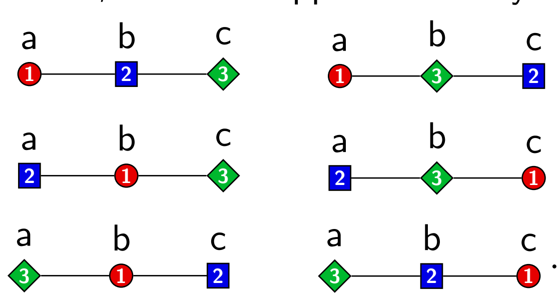
color w/ green (2) and blue (3):

The chromatic polynomial

Let $\chi(G, t)$ be the number of colorings with a palate of t colors.

For example, the path graph $P_3 =$  has

$\chi(P_3, 0) = \chi(P_3, 1) = 0$, and $\chi(P_3, 2) = 2$.
With 3 colors, we break into cases: use all 3 colors, or only use 2.
Using exactly 2 colors, this can happen in $\chi(P_3, 2) * \binom{3}{2} = 6$ ways.
Using exactly 3 colors, this can happen in exactly $3! = 6$ ways:



So $\chi(P_3, 3) = 6 + 6 = 12$.
Note that $\chi(P_3, 4)$ is totally defined by these previous terms!

Let $\chi(G,t)$ be the number of colorings with a palate of t colors.

In general, if a graph G has n vertices, then

$$\chi(G,0), \chi(G,1), \ldots, \chi(G,n)$$

determine $\chi(G,t)$ for $t > n$.

Theorem

For a simple (labeled) graph on n vertices, $\chi(G,t)$ is a polynomial in t of degree n, i.e. for some $a_0, \ldots, a_n$, we have

$$\chi(G,t) = a_0 + a_1 t + \cdots + a_n t^n$$

for all $t \in \mathbb{Z}_{\geqslant 0}$.

Example: computing $\chi(P_3,t)$ in general.
We know $\chi(P_3,t)$ is a degree 3 polynomial, i.e.

$$\chi(P_3,t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

satisfying $\chi(P_3,0) = \chi(P_3,1) = 0$, $\chi(P_3,2) = 2$, and $\chi(P_3,3) = 12$.

Example: computing $\chi(P_3,t)$ in general.
We know $\chi(P_3,t)$ is a degree 3 polynomial, i.e.

$$\chi(P_3,t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

satisfying $\chi(P_3,0) = \chi(P_3,1) = 0$, $\chi(P_3,2) = 2$, and $\chi(P_3,3) = 12$. So we need to solve

$$\begin{aligned} 0 =& \chi(P_3,0) = a_0 \\ 0 =& \chi(P_3,1) = a_0 + a_1 + a_2 + a_3 \\ 2 =& \chi(P_3,2) = a_0 + 2a_1 + 2^2 a_2 + 2^3 a_3 \\ 12 =& \chi(P_3,3) = a_0 + 3a_1 + 3^2 a_2 + 3^3 a_3 \end{aligned}$$

Solving this system gives

$$a_0 = 0, \qquad a_1 = 1, \qquad a_2 = -2, \qquad a_3 = 1.$$

So $\chi(P_3,t) = t - 2t^2 + t^3 = t(t-1)^2$.

You try: Calculate the number of ways to color (a labeled) C_4 with palates of 0, 1, 2, 3, and 4 colors. Then compute $\chi(C_4,t)$. Finally, calculate the number of ways to color C_4 with a palate of 5 colors by counting and verify that your answer matches $\chi(C_4,5)$.

Let $\chi(G,t)$ be the number of colorings with a palate of t colors.

Theorem

For a simple (labeled) graph on n vertices, $\chi(G,t)$ is a polynomial in t of degree n, i.e. for some $a_0, \ldots, a_n$, we have

$$\chi(G,t) = a_0 + a_1 t + \cdots + a_n t^n$$

for all $t \in \mathbb{Z}_{\geqslant 0}$.

Some properties of the $\chi(G,t)$ that can help you error check:

- If $n > 0$, then $\chi(G,0) = 0$. So t is a factor of $\chi(G,t)$.
- If $n > 1$ and G has any edges, then $\chi(G,1) = 0$. So $(t-1)$ is a factor of $\chi(G,t)$.
- Similarly, G cannot be colored using any fewer that $\chi = \chi(G)$ colors, so
  $$\chi(G,0) = \chi(G,1) = \cdots = \chi(G,\chi-1) = 0.$$
  So
  $$t(t-1)(t-2)\cdots(t-(\chi-1))$$
  is a factor of $\chi(G,t)$.

Some properties of the $\chi(G,t)$ that can help you error check:

1. G cannot be colored using any fewer that $\chi = \chi(G)$ colors, so
   $$\chi(G,0) = \chi(G,1) = \cdots = \chi(G,\chi-1) = 0.$$
   So
   $$t(t-1)(t-2)\cdots(t-(\chi-1))$$
   is a factor of $\chi(G,t)$.
2. The coefficient of t^n in $\chi(G,t)$ is 1.
3. The coefficient of t^{n-1} in $\chi(G,t)$ is $-|E|$.
4. The coefficients alternate in signs.
5. If G has connected components C_1, ..., C_ℓ, then
   $$\chi(G,t) = \chi(C_1,t)\chi(C_2,t)\cdots\chi(C_\ell,t).$$
   Namely, the coefficients of t^0, ..., $t^{\ell-1}$ in $\chi(G,t)$ are all zero, and the coefficients of t^ℓ, ..., t^n in $\chi(G,t)$ are all not zero.

For more notes, see
http://en.wikipedia.org/wiki/Chromatic_polynomial.