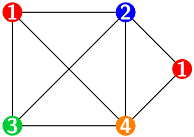
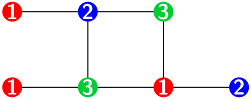
Last time:

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

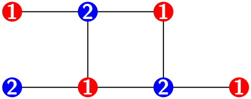


using 4 colors

$\chi = 4$



using 3 colors



using 2 colors

$\chi = 2$

The chromatic number of a graph G, denoted $\chi(G)$, is the least number of colors needed for a coloring of this graph.

To calculate, argue that the graph can't be colored in $\chi - 1$ colors, and then give a coloring with exactly χ colors.

The chromatic polynomial

Question: Given a palate of t colors, how many ways are there to color a (labeled) graph using that palate? (You don't have to use all the colors at once.)

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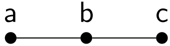
$P_3 =$ 

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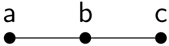
$P_3 =$ 

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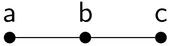
$P_3 =$ 

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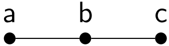
$P_3 =$ 

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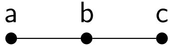
$P_3 =$ 

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With 2 colors (say red and blue), it can be colored in 2 ways:

So $\chi(P_3,2) = 2$.

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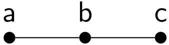
For example, the path graph $P_3 =$ 

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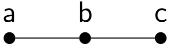
For example, the path graph $P_3 =$ 

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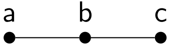
For example, the path graph $P_3 =$ 

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Using exactly 2 colors, this can happen in $\chi(P_3,2) * \binom{3}{2} = 6$ ways:

color w/ red (1) and blue (2): 

color w/ red (1) and green (3): 

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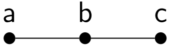
For example, the path graph $P_3 =$ 

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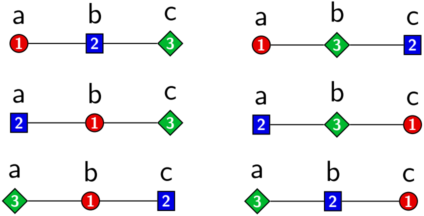

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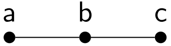
For example, the path graph $P_3 =$ 

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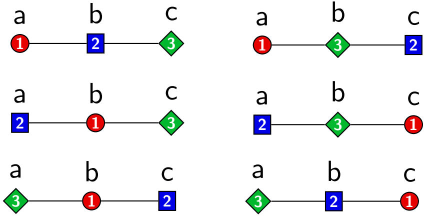

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So $\chi(P_3,3) = 6 + 6 = 12$.

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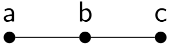
For example, the path graph $P_3 =$ 

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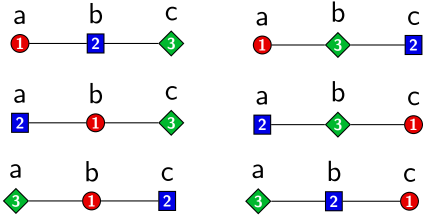

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So $\chi(P_3,3) = 6 + 6 = 12$.
Note that $\chi(P_3,4)$ is totally defined by these previous terms!

Let $\chi(G,t)$ be the number of colorings with a palate of t colors.

In general, if a graph G has n vertices, then

$$\chi(G,0), \chi(G,1), \ldots, \chi(G,n)$$

determine $\chi(G,t)$ for $t > n$.

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Theorem

For a simple (labeled) graph on n vertices, $\chi(G,t)$ is a polynomial in t of degree n, i.e. for some $a_0, \ldots, a_n$, we have

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We know $\chi(P_3, t)$ is a degree 3 polynomial, i.e.

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So $\chi(P_3, t) = t - 2t^2 + t^3 = t(t-1)^2$.

You try: Calculate the number of ways to color (a labeled) C_4 with palates of 0, 1, 2, 3, and 4 colors. Then compute $\chi(C_4, t)$. Finally, calculate the number of ways to color C_4 with a palate of 5 colors by counting and verify that your answer matches $\chi(C_4, 5)$.

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3. The coefficient of t^{n-1} in $\chi(G,t)$ is $-|E|$.

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2. The coefficient of t^n in $\chi(G,t)$ is 1.
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4. The coefficients alternate in signs.

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For more notes, see
http://en.wikipedia.org/wiki/Chromatic_polynomial.