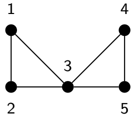
Welcome back warmup

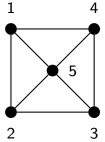
Recall, a graph invariant is a statistic about a graph that is preserved under isomorphisms (relabeling of the vertices).

Before looking through the notes, list as many "graph invariants" as you can from memory. Then, for each, compute that invariant on the following graphs.

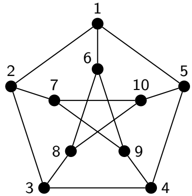
$G_1 =$



$G_2 =$



$G_3 =$



Once you run out, there is a list of graph invariants on the next page–fill out your list and continue computing graph invariants.

Graph invariants

Recall, a graph invariant is a statistic about a graph that is preserved under isomorphisms (relabeling of the vertices).

1. $|V|$, $|E|$
2. Degree sequence

   Also: Minimum degree, maximum degree, vertex of degree d_1 adjacent to vertex of degree d_2, ...
3. Bipartite or not

   If any subgraph is not bipartite, then G is not bipartite. A graph is bipartite if and only if it has no odd cycles as subgraphs.
4. Paths or cycles of particular lengths

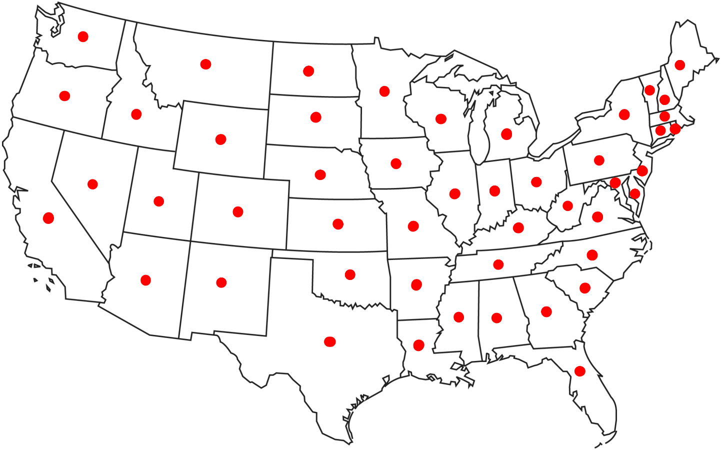
   Also: longest path or cycle length, maximal paths of certain lengths, ...
5. Edge connectivity $\lambda(G)$ and vertex connectivity $\kappa(G)$.
6. Does it have an Euler trail/circuit?
7. Does it have a Hamilton path/cycle?

Maps coloring

Encode information about which regions on a map share a border.

Maps coloring

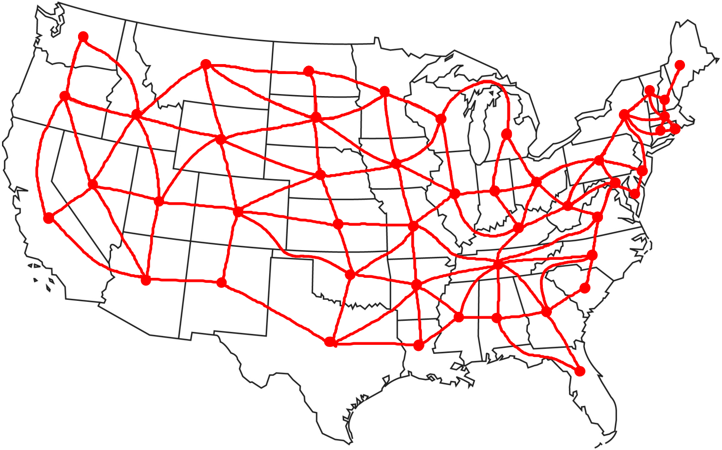
Encode information about which regions on a map share a border.



Mark each region with a vertex.

Maps coloring

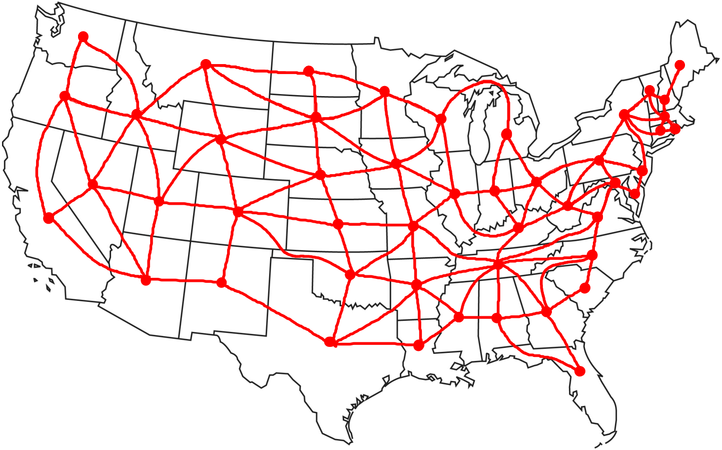
Encode information about which regions on a map share a border.



Mark each region with a vertex. Draw an edge between to vertices if the corresponding regions share a boarder.

Maps coloring

Encode information about which regions on a map share a border.

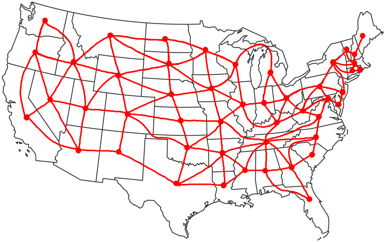


Mark each region with a vertex. Draw an edge between to vertices if the corresponding regions share a boarder.

Map coloring problems: color a map so that no two adjacent regions get the same color.

Maps coloring

Encode information about which regions on a map share a border.



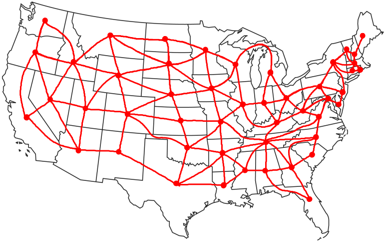
Mark each region with a vertex. Draw an edge between to vertices if the corresponding regions share a boarder.

Map coloring problems: color a map so that no two adjacent regions get the same color.

Four color problem: In order to color the vertices of a plane map so that no two adjacent vertices get the same color, you will need no more than four colors.

Maps coloring

Encode information about which regions on a map share a border.



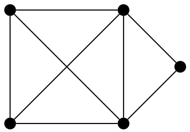
Mark each region with a vertex. Draw an edge between to vertices if the corresponding regions share a boarder.

Map coloring problems: color a map so that no two adjacent regions get the same color.

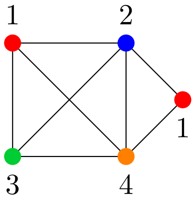
Four color problem: In order to color the vertices of a plane map so that no two adjacent vertices get the same color, you will need no more than four colors. (No elementary proof!)

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

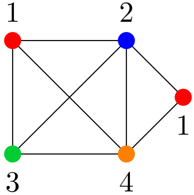


A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

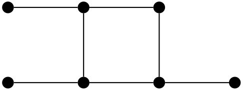


using 4 colors

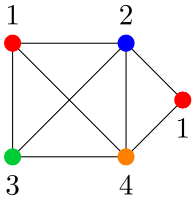
A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.



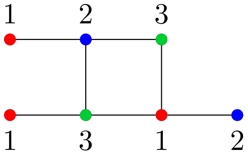
using 4 colors



A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

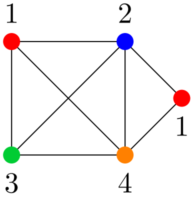


using 4 colors

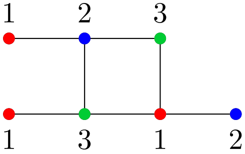


using 3 colors

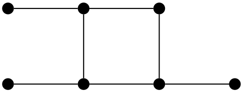
A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.



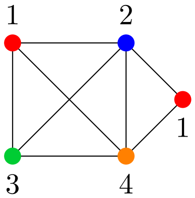
using 4 colors



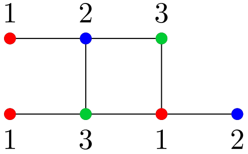
using 3 colors



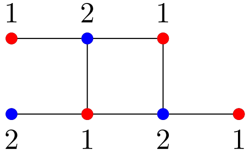
A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.



using 4 colors

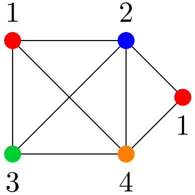


using 3 colors

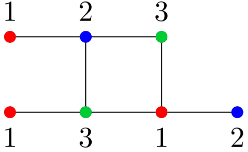


using 2 colors

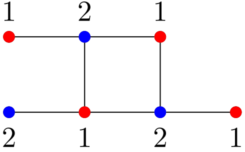
A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.



using 4 colors



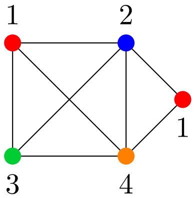
using 3 colors



using 2 colors

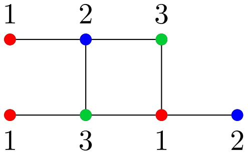
The chromatic number of a graph G, denoted $\chi(G)$, is the least number of colors needed for a coloring of this graph.

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

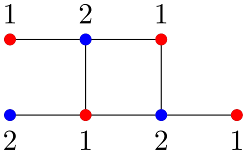


using 4 colors

$\chi = 4$



using 3 colors



using 2 colors

$\chi = 2$

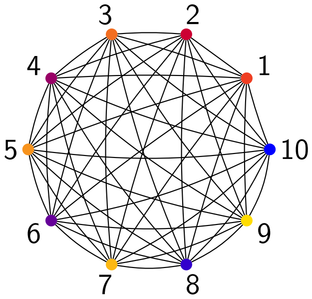
The chromatic number of a graph G, denoted $\chi(G)$, is the least number of colors needed for a coloring of this graph.

To calculate, argue that the graph can't be colored in $\chi - 1$ colors, and then give a coloring with exactly χ colors.

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color. The chromatic number of a graph G, denoted $\chi(G)$, is the least number of colors needed for a coloring of this graph.

Lemma

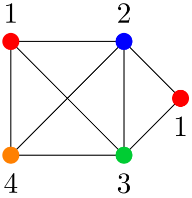
The complete graph on n vertices can only be colored in exactly n colors. Namely $\chi(K_n) = n$.



Cliques

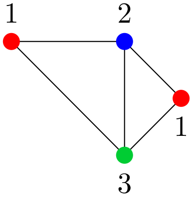
Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:

$G =$



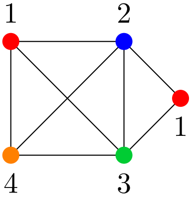
$\rightarrow$

$H =$

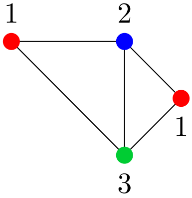


Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:

$G =$ 

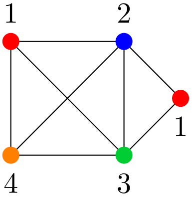
$\rightarrow$

$H =$ 

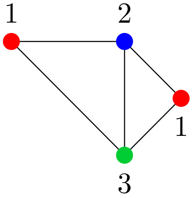
So $\boxed{\chi(G) \geqslant \chi(H)}$.

Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:

$G =$ 

$\rightarrow$

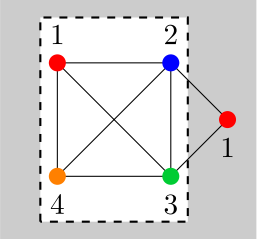
$H =$ 

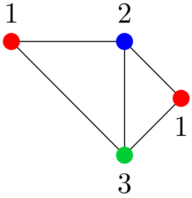
So $\boxed{\chi(G) \geqslant \chi(H)}$.

Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:

$G =$ 

$\rightarrow$ $H =$ 

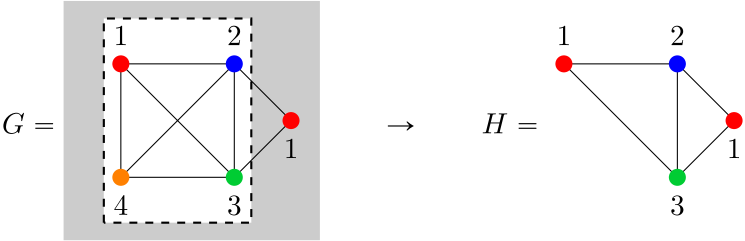
So $\boxed{\chi(G) \geqslant \chi(H)}$.

Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

Ex: the G has K_4 as a subgraph, so $\chi(G) \geqslant \chi(K_4) = 4$.

Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:



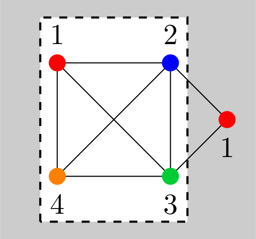
So $\boxed{\chi(G) \geqslant \chi(H)}$.

Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

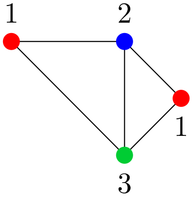
Ex: the G has K_4 as a subgraph, so $\chi(G) \geqslant \chi(K_4) = 4$. And since we gave a 4-coloring, we know $\chi(G) \leqslant 4$.

Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:

$G =$ 

$\rightarrow$

$H =$ 

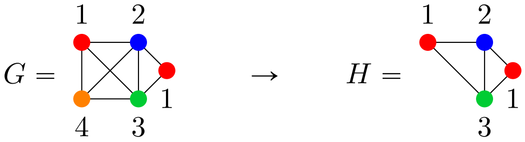
So $\boxed{\chi(G) \geqslant \chi(H)}$.

Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

Ex: the G has K_4 as a subgraph, so $\chi(G) \geqslant \chi(K_4) = 4$. And since we gave a 4-coloring, we know $\chi(G) \leqslant 4$. So $\boxed{\chi(G) = 4}$ exactly.

Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:



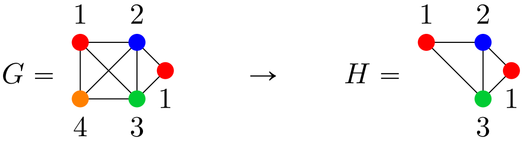
So $\boxed{\chi(H) \leqslant \chi(G)}$.

Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique.

Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:



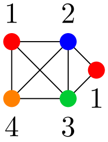
So $\boxed{\chi(H) \leqslant \chi(G)}$.

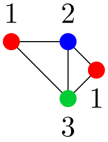
Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique.

Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:

$G =$ 

$\rightarrow$ $H =$ 

So $\boxed{\chi(H) \leqslant \chi(G)}$.

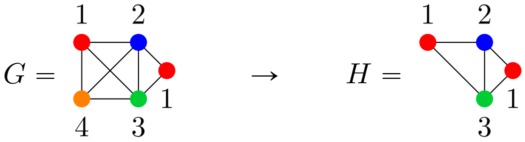
Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique.

Ex: $\omega(G) = 4$, $\omega(H) = 3$.

Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:



So $\boxed{\chi(H) \leqslant \chi(G)}$.

Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique.

Ex: $\omega(G) = 4$, $\omega(H) = 3$.

Then

So $\boxed{\omega(G) \leqslant \chi(G)}$.

(Both chromatic and clique numbers are graph invariants.)

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

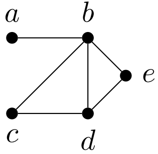
We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.

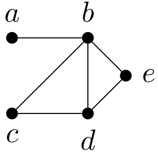
Ex: In

$G =$  ,

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.

Ex: In

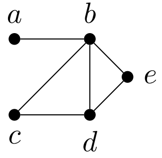
$G =$  ,

$\{a, d\}$ is an independent set;

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.

Ex: In

$G =$  ,

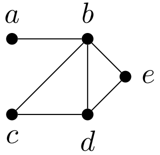
$\{a, d\}$ is an independent set; so is $\{c, e\}$;

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.

Ex: In

$G =$



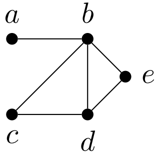
,

$\{a, d\}$ is an independent set; so is $\{c, e\}$; so is $\{a, c, e\}$.

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.

Ex: In

$G =$ 

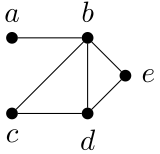
,

$\{a, d\}$ is an independent set; so is $\{c, e\}$; so is $\{a, c, e\}$. But $\{a, d, e\}$ is not, since d and e share an edge.

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.

Ex: In

$G =$  ,

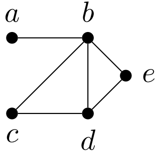
$\{a, d\}$ is an independent set; so is $\{c, e\}$; so is $\{a, c, e\}$. But $\{a, d, e\}$ is not, since d and e share an edge.

The independence number $\alpha(G)$ is the size of a largest independent set. (Yet another graph invariant!)

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.

Ex: In

$G =$ 

, $\boxed{\alpha(H) = 3}$

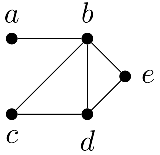
$\{a, d\}$ is an independent set; so is $\{c, e\}$; so is $\{a, c, e\}$. But $\{a, d, e\}$ is not, since d and e share an edge.

The independence number $\alpha(G)$ is the size of a largest independent set. (Yet another graph invariant!)

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.

Ex: In

$G =$ , $\boxed{\alpha(H) = 3}$

$\{a, d\}$ is an independent set; so is $\{c, e\}$; so is $\{a, c, e\}$. But $\{a, d, e\}$ is not, since d and e share an edge.

The independence number $\alpha(G)$ is the size of a largest independent set. (Yet another graph invariant!)

Note: $\alpha(G) = \omega(\bar{G})$

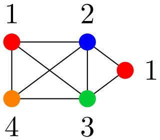
(Recall $\bar{G}$ is the *complement* of G — toggle all of the edges.)

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them. The independence number $\alpha(G)$ is the size of a largest independent set. (Yet another graph invariant!)

Note: $\alpha(G) = \omega(\bar{G})$
(Recall $\bar{G}$ is the *complement* of G — toggle all of the edges.)

Also, in a coloring of a graph, the vertices of any given color are all independent.

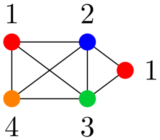


We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them. The independence number $\alpha(G)$ is the size of a largest independent set. (Yet another graph invariant!)

Note: $\alpha(G) = \omega(\bar{G})$
(Recall $\bar{G}$ is the *complement* of G — toggle all of the edges.)

Also, in a coloring of a graph, the vertices of any given color are all independent.



So $\boxed{\chi(G) \geqslant |V|/\alpha(G) \text{ and } \chi(G) \geqslant \omega(G)}$

(Dividing $|V|$ vertices evenly into sets of size $\alpha(G)$ gets you $|V|/\alpha(G)$ sets.)