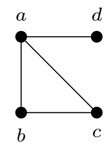
Math 365 – Monday 4/15/19 – 10.3 & 10.4 Subgraphs, connectivity, and walks

Exercise 49. (a) Consider the graph

$G =$ 

(i) Give an example of a subgraph of G that is not induced.

(ii) How many induced subgraphs does G have? List them.

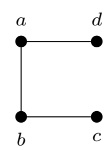
(iii) How many subgraphs does G have?

(iv) Let e be the edge connecting a and d. Draw $G - e$, G/e, and $(G/e)_{\text{simple}}$.

(v) Let e be the edge connecting a and c. Draw $G - e$, G/e, and $(G/e)_{\text{simple}}$.

(vi) Draw $\bar{G}$.

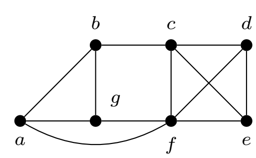
(b) Show that

$G =$ 

is isomorphic to its complement.

(c) Find a simple graph with 5 vertices that is isomorphic to its own complement. (Start with: how many edges must it have?)

Exercise 50. Let G be the graph



Note that G is simple, so that walks in G are determined by sequences of vertices.

(a) Decide which of the following sequences of vertices determine walks. For those that are walks, decide whether they can be classified more finely (is it a circuit, a path, a cycle, or a a trail?).

(i) a, b, g, f, c, b
(ii) b, g, f, c, b, g, a
(iii) c, e, f, c
(iv) c, e, f, c, e
(v) a, b, f, a
(vi) f, d, e, c, b

(b) Verify that the following sequences of vertices determine paths. Decide whether they are maximal. If not, extend the sequence to a maximal path.

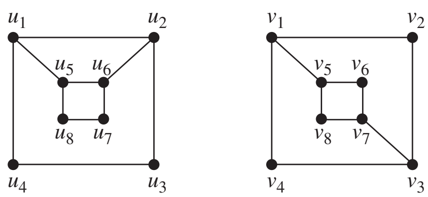
(i) a, b, g, f, d (ii) $d, e, f, c, b, a, g,$ (iii) f, a, g, b, c

(c) Give an example of a maximal path whose length is not maximal (a path that cannot be extended, but which is shorter than some other path in G).

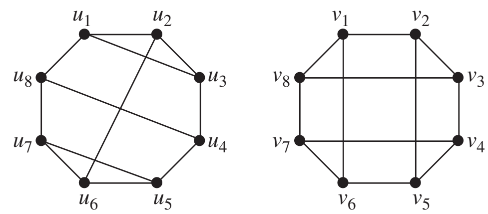
Exercise 51.

(a) For each pair of graphs, either use paths or cycles to show that they are not isomorphic, or give an isomorphism.

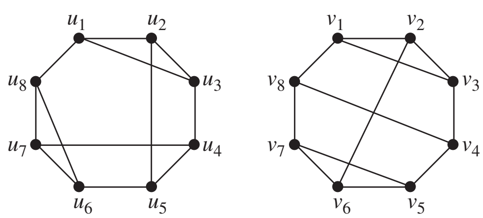
(i)



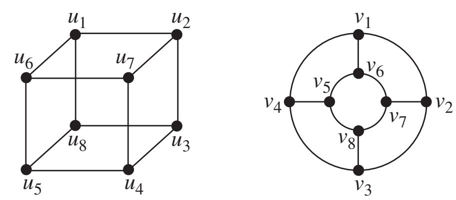
(ii)



(iii)

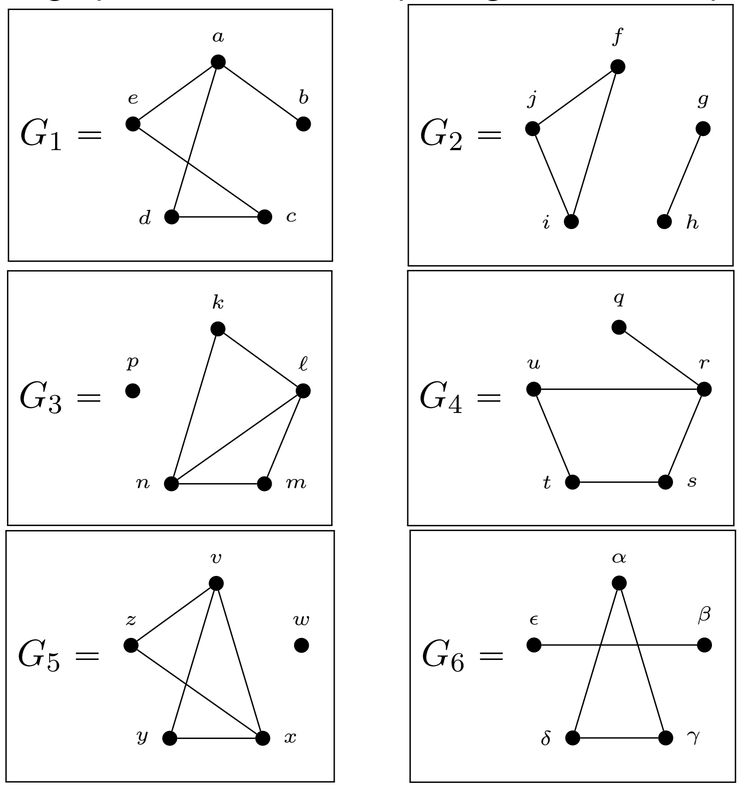


(iv)



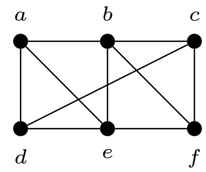
(b) Explain why if v is a vertex of odd degree, then there is a walk from v to another vertex of odd degree.

Warmup: Find which pairs of the following graphs are isomorphic. For any two graphs that are isomorphic, give an isomorphism.

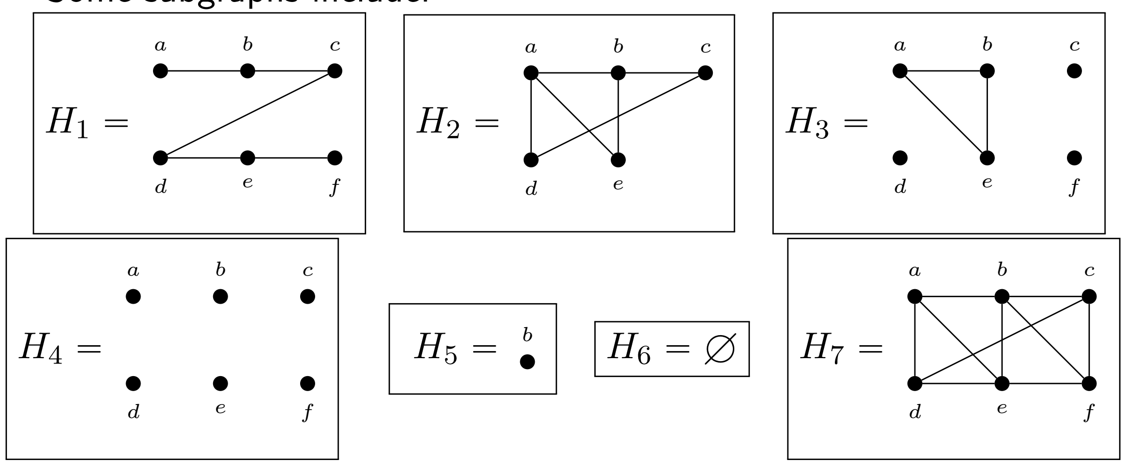


New graphs from old

Let $G = (V, E)$ be a simple graph. A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ such that $W \subseteq V$ and $F \subseteq E$. For example, let

$G =$ 

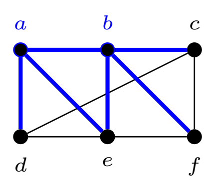
Some subgraphs include:



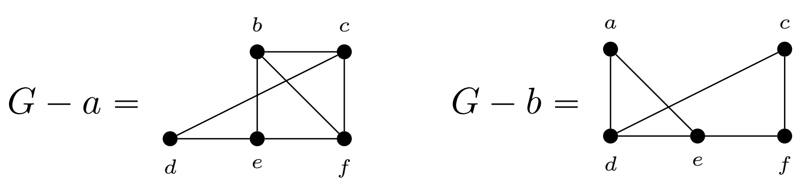
New graphs from old

Let $G = (V, E)$ be a simple graph.

For $v \in V$, the graph $G - v$ is the graph made by deleting v and any edge incident to v. For example, if

$G =$ 

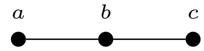
then



Let $W \subseteq V$. The subgraph induced by W, denoted $G[W]$, is the subgraph made by deleting everything not in W. For example, $G[\{b, c, d, e, f\}] = G - a$.

Counting subgraphs

Step 1: Break into cases based on the vertex set.

For example, let $G =$ 

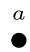
G hase vertex set $V = V_G = \{a, b, c\}$.

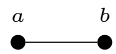
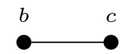
If $H \subseteq G$, then $V_H \subseteq \{a, b, c\}$. Possibilities:

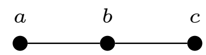
$$\varnothing, \quad \{a\}, \quad \{b\}, \quad \{c\}, \quad \{a, b\},$$
$$\{a, c\}, \quad \{b, c\}, \quad \text{or} \quad \{a, b, c\}.$$

Step 2: Draw the induced graphs for each vertex subset.

In our example,

$G[\varnothing] = \varnothing$, $G[\{a\}] =$ , $G[\{b\}] =$ , $G[\{c\}] =$ ,

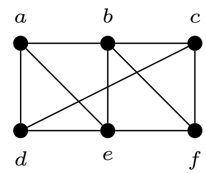
$G[\{a, b\}] =$ , $G[\{a, c\}] =$ , $G[\{b, c\}] =$ 

$G[\{a, b, c\}] = G =$ .

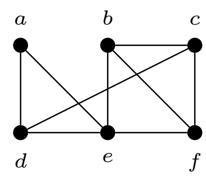
Step 3: Count the number of subgraphs of the induced graphs that have the same vertex set, but possibly fewer edges. This reduces to looking at each edge and deciding to keep it or lose it. So there are $2^{\#\text{edges}}$ such subgraphs. $1 + 1 + 1 + 1 + 2 + 1 + 2 + 2^2 = \boxed{13}$

Edge operations.

Subtraction: Let $\epsilon \in E$. Then $G - \epsilon$ is the subgraph of G with vertex set V and edge set $E - \{\epsilon\}$.
For example, if

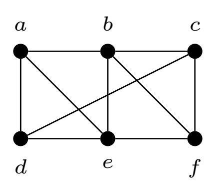
$G =$ 

then

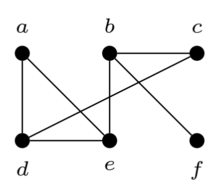
$G - (a{-}b) =$ 

Edge operations.

If $F \subseteq E$, the graph G is the subgraph with vertex set V and edge set $E - F$.
For example, if

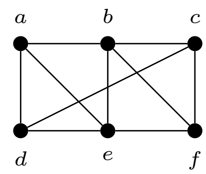
$G =$ 

then

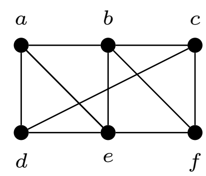
$G - \{a{-}b, c{-}f, e{-}f\} =$ 

Edge operations.

Addition: For an edge ϵ on the vertex set V but not in E, $G + \epsilon$ is the graph containing G satisfying $(G + \epsilon) - \epsilon = G$. For example, if

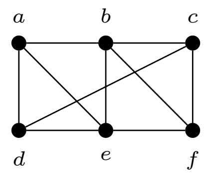
$G =$ 

then

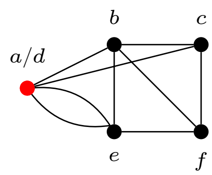
$G + (a{-}e) =$ 

Edge operations.

Let $\epsilon \in E$. There are two kinds of "contraction" of ϵ: contraction of graphs (allowing for multiple edges and loops) and contraction of simple graphs (not allowing for multiple edges and loops).
Contraction of an edge in graphs: If we're considering all graphs, then the graph G/ϵ is the graph obtained by contracting the edge ϵ, which means merging the vertices that are incident to ϵ.
For example, if

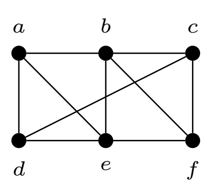
$G =$ 

and $\epsilon = a{-}d$, then

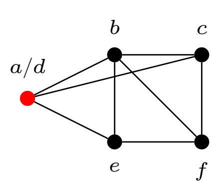
$G/\epsilon =$ 

Edge operations.

Let $\epsilon \in E$. There are two kinds of "contraction" of ϵ: contraction of graphs (allowing for multiple edges and loops) and contraction of simple graphs (not allowing for multiple edges and loops).
Contraction of an edge in simple graphs: If we're considering only simple graphs, then the graph G/ϵ is the graph obtained by contracting the edge ϵ, and then deleting any loops or multiple edges.
For example, if

$G =$ 

and $\epsilon = a{-}d$, then

$(G/\epsilon)_{\text{simple}} =$ 

Note that G/e and $(G/\epsilon)_{\text{simple}}$ are not in general subgraphs of G.

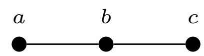
Unions

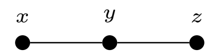
The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2).$$

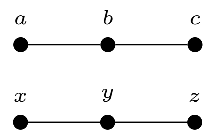
Examples:

If

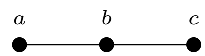
$G_1 =$ 

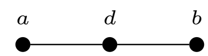
and $G_2 =$ 

then

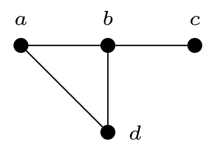
$G_1 \cup G_2 =$ 

If

$G_1 =$ 

and $G_2 =$ 

then

$G_1 \cup G_2 =$ 

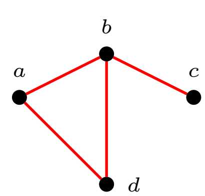
Complements

Consider G as a subgraph of $K[V]$, the complete graph on the vertex set V. The complement of the graph G is

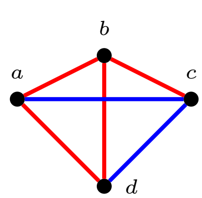
$$\bar{G} = (V, E_{K[V]} - E).$$

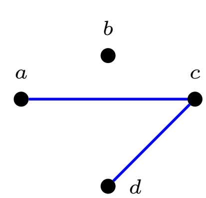
In other words, G and $\bar{G}$ have the same vertex set, but u and v are adjacent in $\bar{G}$ if and only if u and v are not adjacent in G.

Example: Let

$G =$ 

Then

$K[V] =$ 

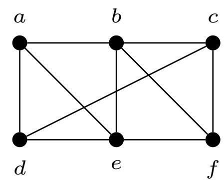
so $\overline{G} =$ 

Connectedness

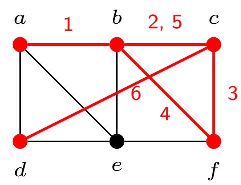
Let $G = (V, E)$ be a graph. A walk is an alternating sequence of vertices and edges

$$w = (v_0, e_1, v_1, e_2, \cdots, e_n, v_n)$$

such that e_i has endpoints v_{i-1} and v_i. We say w has length n. For example, if

$G =$  ,

the walk $(a, a{-}b, b, b{-}c, c, c{-}f, f, f{-}b, b, b{-}c, c, c{-}d, d)$ looks like



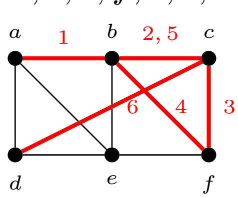
Special kinds of walks:

1. A closed walk or circuit is a walk where $v_0 = v_n$.
2. A path is a walk so that no vertices (and therefore edges) are repeated.
3. A cycle is a walk where $v_0 = v_n$ but no other vertices are repeated.
4. A trail is a walk where no edges are repeated.

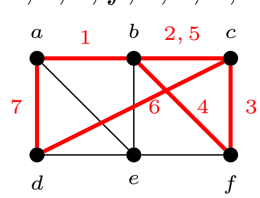
Note: See the remark on p. 679 of the book to reconcile the difference between the terminology we're using and the terminology in the book!!

In a simple graph, the sequence of vertices determines the walk, since there's at most one edge between any two vertices.

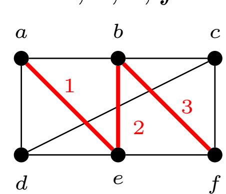
Walk:
a, b, c, f, b, c, d



Circuit:
a, b, c, f, b, c, d, a



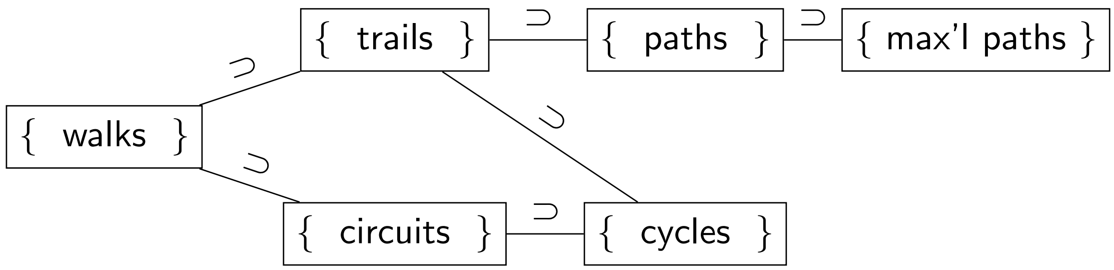
Path:
a, e, b, f



Special kinds of walks:

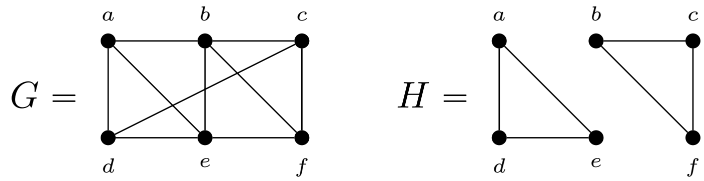
1. A closed walk or circuit is a walk where $v_0 = v_n$.
2. A path is a walk so that no vertices (and therefore edges) are repeated.
3. A cycle is a walk where $v_0 = v_n$ but no other vertices are repeated.
4. A trail is a walk where no edges are repeated.

Note:



A maximal path is a path that cannot be extended on either end to be a longer path.

A graph is connected if for every pair of vertices u and v, there is a walk from u to v. For example:



G is connected; H is not.

A connected component of a graph is a maximally connected subgraph of G (H above has two connected components). Note that every connected component (or union of connected components) is an induced subgraph. Further, if W is the set of vertices in some connected component, then the induced subgraph H by W has the property that the degrees of the vertices in H are the same as the degrees of the corresponding vertices in G.

We say two vertices are connected if there is a walk between them.

Connected is an equivalence relation on vertices:

- Reflexive: The walk from v to itself is the walk of length 0: $v_0 = v = v_n$.
- Symmetric: If there is a walk from u to v, then the walk from v to u is the reverse sequence.
- Transitive: If there is a walk from a to b $w_a = a, e_1, v_1, \ldots, e_n, b$, and a walk from b to c, $w_b = b, e'_1, v'_1, \ldots, e'_n, c$, then

$$w = a, e_1, v_1, \ldots, e_n, b, e'_1, v'_1, \ldots, e'_n, c$$

is a walk from a to c.

The equivalence class is the connected component.

Graph invariants

Recall, a graph invariant is a statistic about a graph that is preserved under isomorphisms (relabeling of the vertices). Namely, if you don't need the labels to calculate the statistic, then it's probably a graph invariant.

1. $|V|$, $|E|$
2. Degree sequence

   Also: Minimum degree, maximum degree, vertex of degree d_1 adjacent to vertex of degree d_2, ...
3. Bipartite or not

   If any subgraph is not bipartite, then G is not bipartite. A graph is bipartite if and only if it has no odd cycles as subgraphs.
4. Connected or not
5. Paths or cycles of particular lengths

   Also: longest path or cycle length, maximal paths of certain lengths, ...