Math 365 – Wednesday 4/10/19 – 10.1 & 10.2 Graphs

Exercise 44. (Relations and digraphs) For each the relations in Exercise 43(a), draw the corresponding directed graph where $V = \{0, 1, 2, 3\}$ and

$$a \to b \qquad \text{if} \qquad a \sim b.$$

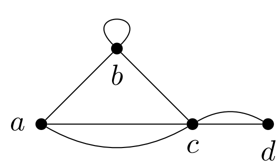
What properties of the directed graphs correspond to the symmetric, reflexive, and transitive properties of the corresponding relations? For the digraphs corresponding to equivalence relations, what do the equivalence classes look like?

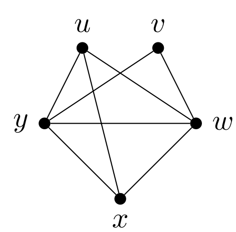
Exercise 45. (Applied graphs) Pick four of the Examples 2–12 in Section 10.1, and quickly summarize them. What is V? What is E? And what kind of graph results? For example, in Example 1,

$$V = \{ \text{ people } \}$$
$$E = \{a{-}b \mid a \neq b, \text{ and } a \text{ and } b \text{ are acquainted } \}$$

The resulting graph is simple.

Exercise 46. Let

$G =$ 

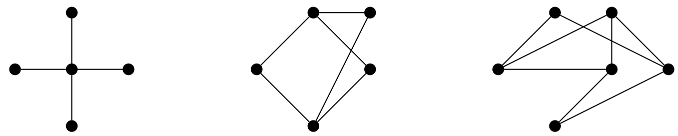
$H =$ 

(a) In G, what is the neighborhood of a? What is the neighborhood of b?
(b) Calculate the degrees of each vertex in G and H.
(c) Verify the handshake theorem on G and H.

Exercise 47.

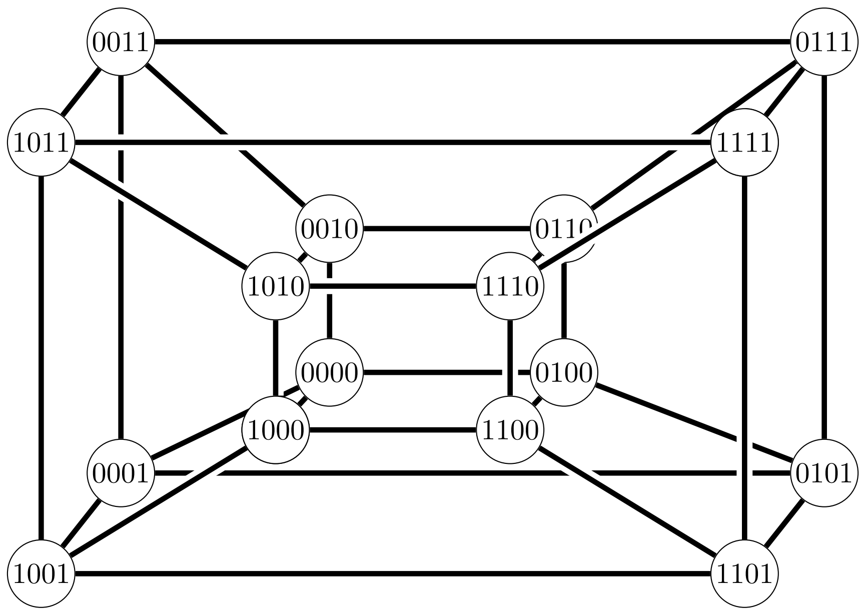
(a) Draw C_6, W_6 K_6, and $K_{5,3}$.

(b) Which of the following are bipartite? Justify your answer.



(c) Hypercubes are bipartite.

(i) The following is the 4-cube:



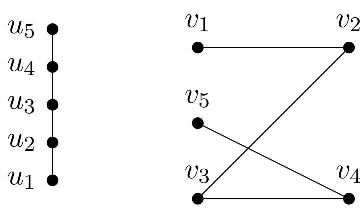
Shade in the vertices that have an even number of 0's. Explain why the 4-cube is bipartite.

(ii) Explain why Q_n is bipartite in general.
[Hint: Show that a vertex with an even (respectively, odd) number of 0's will never be adjacent to another vertex with an even (respectively, odd) number of 0's.]

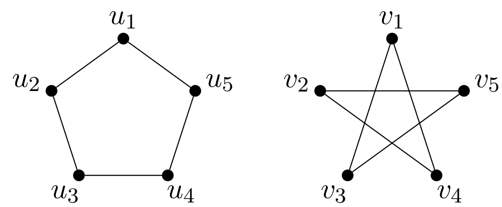
Exercise 48.

(a) For each of the following pairs of graphs, first list their degree sequences. Then decide whether they are isomorphic or not. If not, say why. If they are, give a bijection on the vertices that preserves the edges, and draw the unlabeled graph that represents the corresponding isomorphism class of graphs.

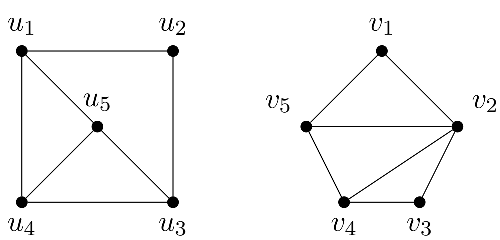
(i)



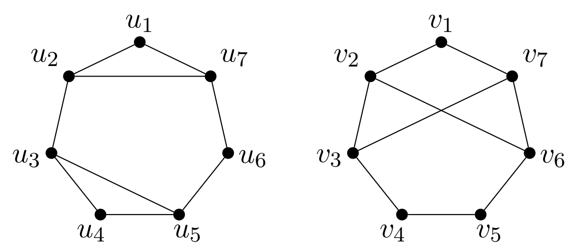
(ii)



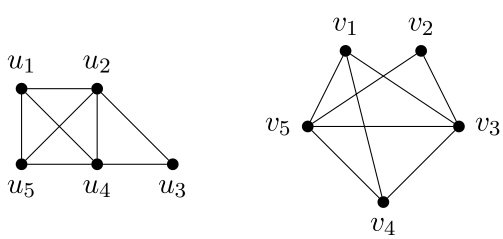
(iii)



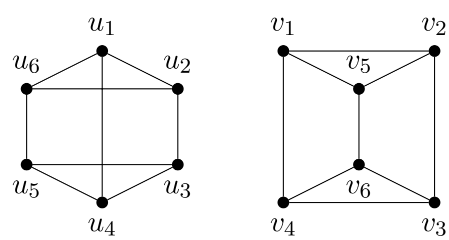
(iv)



(v)



(vi)



(b) How many isomorphism classes are there of simple graphs with 4 vertices? Draw them.

(c) How many edges does a graph have if its degree sequence is $4, 3, 3, 2, 2$? Draw a graph with this degree sequence. Can you draw a simple graph with this sequence?

(d) For which values of n, m are these graphs regular? What is the degree?

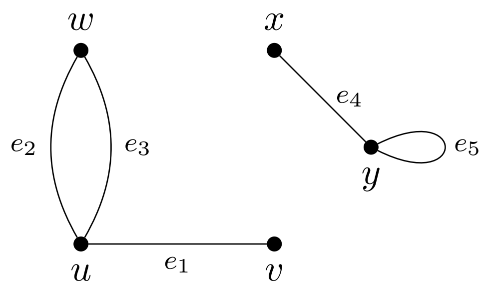
(i) K_n (ii) C_n (iii) W_n (iv) Q_n (v) $K_{m,n}$

(e) How many vertices does a regular graph of degree four with 10 edges have?

(f) Show that isomorphism of simple graphs is an equivalence relation.

A graph is a set of objects, or vertices, together with a (multi)set of edges that connect pairs of vertices. (Think driving routes between cities, or social connections between people.)

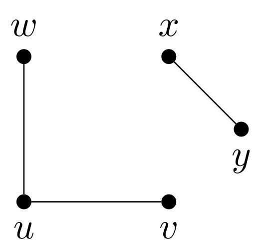
Example:



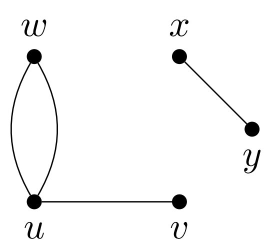
Here, the vertices are $V = \{u, v, w, x, y\}$, and the edges are $E = \{e_1 = u{-}v,\ e_2 = u{-}w,\ e_3 = u{-}w,\ e_4 = x{-}y,\ e_5 = y{-}y\}$. An edge that connects a vertex to itself (like e_5) is called a loop. We say a vertex a is adjacent to a vertex b if there is an edge connecting a and b. (Notice that for a generic graph, "adjacency" is a symmetric relation, but is not reflexive nor is it transitive.)

Classes of graphs:

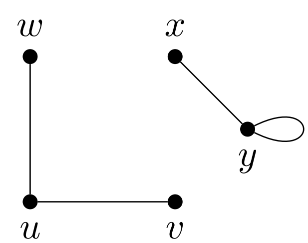
A graph is simple if there are no loops and every pair of vertices has at most one edge between them.



Simple!



NOT simple!

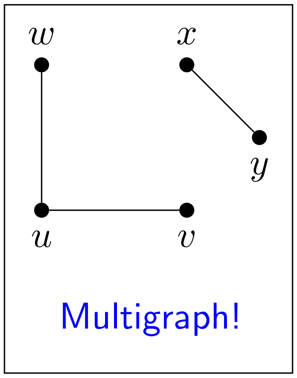


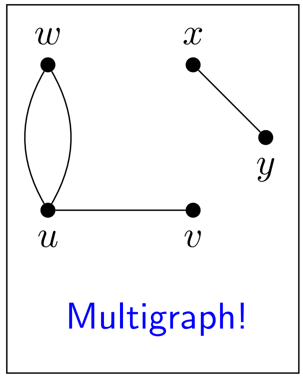
NOT simple!

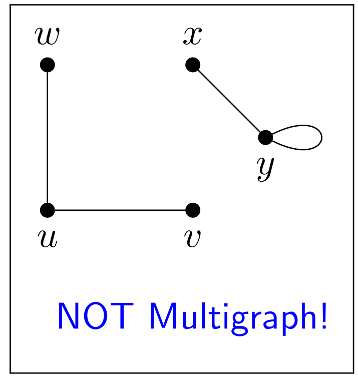
Classes of graphs:

A graph is simple if there are no loops and every pair of vertices has at most one edge between them.

A graph is a multigraph if there are no loops, but there could be multiple edges between two vertices.







Classes of graphs:

A graph is simple if there are no loops and every pair of vertices has at most one edge between them.

A graph is a multigraph if there are no loops, but there could be multiple edges between two vertices.

A graph is a pseudograph if there could be loops or multiple edges. (This is just what <u>we</u> call a graph.)

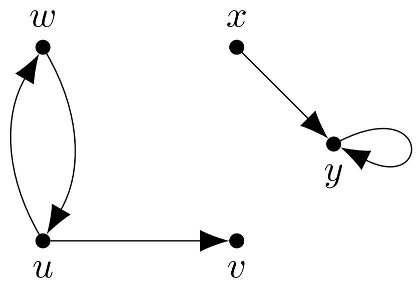
So

$$\{ \text{ pseudographs/graphs } \} \supsetneq \{ \text{ multigraphs } \} \supsetneq \{ \text{ simple graphs } \}.$$

(Note: The $\supsetneq$ symbol is used here because, for example, every simple graph is a multigraph, but there are multigraphs that are not simple.)

Directed graphs

A directed graph (also called a digraph or a quiver) is a graph, together with a choice of direction for each edge. (Think flights from one city to the other, or a flow chart.) For example,



Directed graphs

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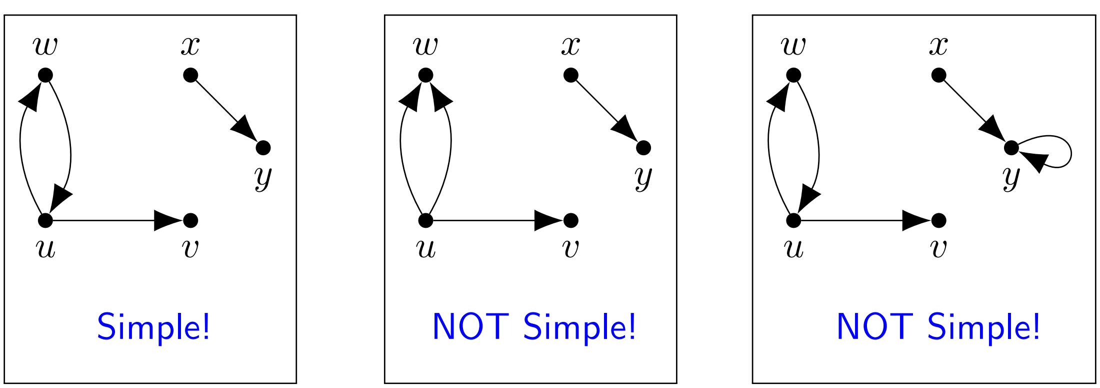
A directed graph is simple if there are no loops and every pair of vertices has at most one edge in each direction between them.



Directed graphs

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A directed graph is simple if there are no loops and every pair of vertices has at most one edge in each direction between them.

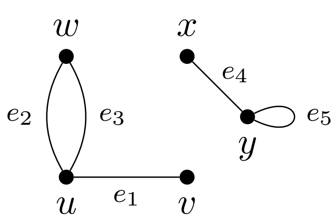
A directed graph is a directed multigraph if there could be loops or multiple edges. (This is just what we call a directed graph)

So

$$\{ \text{ directed (multi)graphs } \} \supsetneq \{ \text{ directed simple graphs } \}.$$

The book also talks about mixed graphs, where some of the edges are directed and some aren't. We usually take care of this by modeling the non-directed edges with *two directed edges*, one in each direction.

$G =$



$N(u) = \{v, w\}$
$N(v) = \{u\}$
$N(w) = \{u\}$
$N(x) = \{y\}$
$N(y) = \{x, y\}$

$N(\{u, v\})$
$= \{u, v, w\}$
$N(\{u, y\})$
$= \{v, w, x, y\}$

We say a vertex a is adjacent to a vertex b if there is an edge connecting a and b. For example, in G,

u is adjacent to w and v; v is adjacent to u;

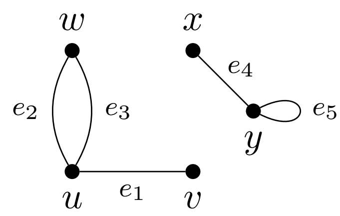
y is adjacent to x and y.

We say that an edge is incident to a vertex if the edge connects to the vertex. For example, in G,

e_1 is incident to u and v; e_5 is incident to y.

If two vertices u and v are adjacent, we we say that they are neighbors, and that u is in the neighborhood $N(v)$ of v (and vice-versa). If $A \subseteq V$, then

$$N(A) = \bigcup_{v \in A} N(v).$$

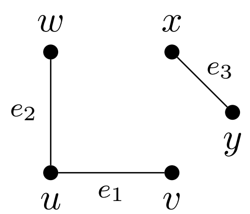
$G =$ 

$N(u) = \{v, w\}$	$\deg(u) = 3$
$N(v) = \{u\}$	$\deg(v) = 1$
$N(w) = \{u\}$	$\deg(w) = 2$
$N(x) = \{y\}$	$\deg(x) = 1$
$N(y) = \{x, y\}$	$\deg(y) = 3$

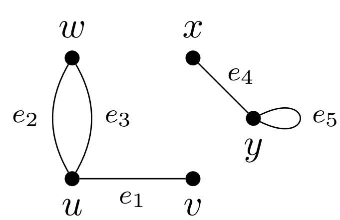
The degree $\deg(v)$ of a vertex v is the number of edge ends attached to v.

Fact: $\deg(v) \geqslant |N(v)|$; and a graph is simple if and only if

$$\deg(v) = |N(v)| \text{ for all } v \in V.$$

$H =$ 

$N(u) = \{v, w\}$	$\deg(u) = 2$
$N(v) = \{u\}$	$\deg(v) = 1$
$N(w) = \{u\}$	$\deg(w) = 1$
$N(x) = \{y\}$	$\deg(x) = 1$
$N(y) = \{x\}$	$\deg(y) = 1$

$G =$ 

$N(u) = \{v, w\}$	$\deg(u) = 3$
$N(v) = \{u\}$	$\deg(v) = 1$
$N(w) = \{u\}$	$\deg(w) = 2$
$N(x) = \{y\}$	$\deg(x) = 1$
$N(y) = \{x, y\}$	$\deg(y) = 3$

The degree $\deg(v)$ of a vertex v is the number of edge ends attached to v. We call a graph regular if all the vertices have the same degree.

Theorem (The handshake theorem)

In a graph $G = (V, E)$,

$$2|E| = \sum_{v \in V} \deg v.$$

Corollary

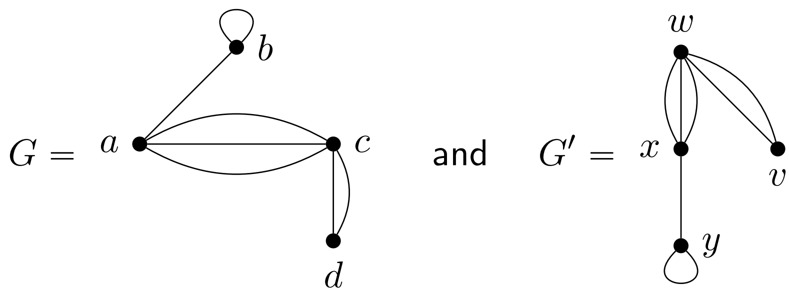
In any graph, there are an even number of odd vertices.

Graph isomorphisms

We say two graphs G and G' are isomorphic if there is a relabeling of the vertices of G that transforms it into G'. In other words, there is a bijection

$$f : V \to V'$$

such that the induced map on E is a bijection $f : E \to E'$.
For example,



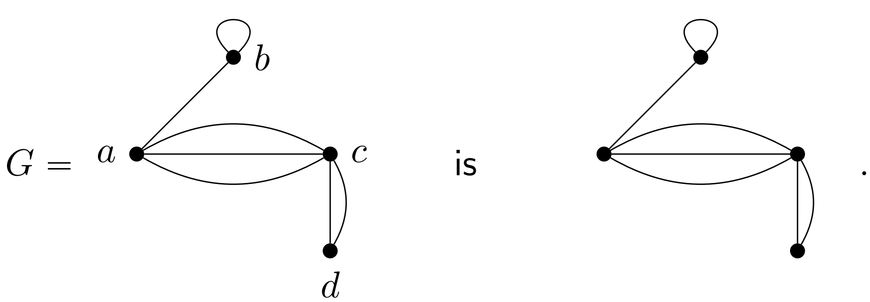
are isomorphic via the map (doesn't depend on the drawing)

$$a \mapsto x, \quad b \mapsto y, \quad c \mapsto x, \quad d \mapsto v.$$

Recall: an equivalence relation on a set $\mathcal{A}$ is a pairing $\sim$ that is reflexive ($a \sim a$), symmetric ($a \sim b$ iff $b \sim a$), and transitive ($a \sim b$ and $b \sim c$ implies $a \sim c$). Given an equivalence relation, an equivalence class is a maximal set of things that are pairwise equivalent. Here, if $\mathcal{G}$ is the set of all graphs, then

$G \sim H$ whenever G is isomorphic to H

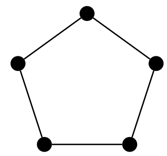
is an equivalence relation. For an equivalence class of graphs, we draw the associated unlabeled graph. For example, the equivalence class of graphs corresponding to



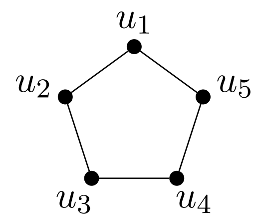
Special graphs

Cycles. A cycle C_n is the equivalence class of simple graphs on n vertices $\{v_1, v_2, \ldots, v_n\}$ so that v_i is adjacent to $v_{i\pm1}$ (v_1 is adjacent to v_n).

equivalence class C_5

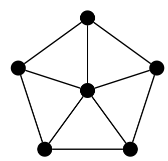


one graph in the class C_5

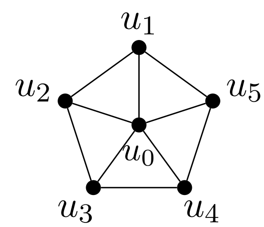


Wheels. A wheel W_n is the cycle C_n together with an additional vertex that is adjacent to every other vertex.

equivalence class W_5

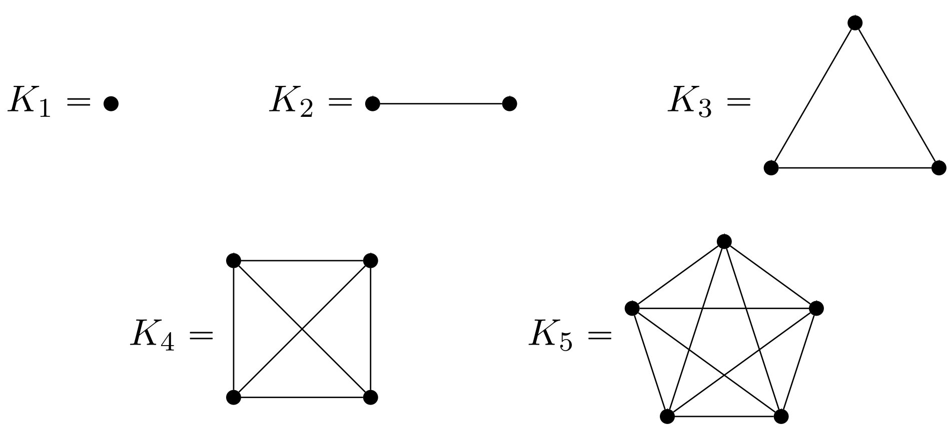


one graph in the class W_5



Special graphs

Complete graphs. The complete graph on n vertices, denoted K_n, is the equivalence class of simple graphs on n vertices so that $N(v) = V - \{v\}$ for all all $v \in V$. For example,

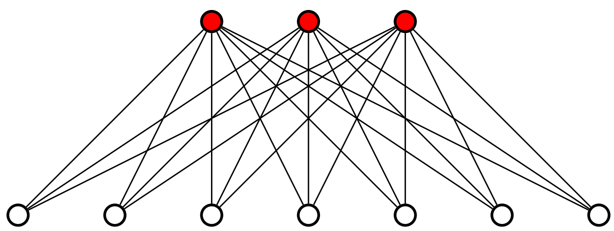


Bipartite graphs. A graph is bipartite if V can be partitioned into two nonempty subsets V_1 and V_2 so that no vertex in V_i is adjacent to any other vertex in V_i for $i = 1$ or 2.
In particular, for any $m \geqslant n \geqslant 1$, the complete bipartite graph $K_{n,m}$ is the class of simple graphs corresponding to the graph with vertices $V = V_1 \cup V_2$, where

$$V_1 = \{v_1, \ldots, v_n\} \quad V_2 = \{u_1, \cdots u_m\}$$

$$N(v_i) = V_2 \qquad \text{and} \qquad N(u_i) = V_1$$

for all i. For example,

$K_{7,3} =$ 

One way to show that a graph is bipartite is to "color" the vertices two different colors, so that no two vertices of the same color are adjacent.

Bipartite graphs. A graph is bipartite if V can be partitioned into two nonempty subsets V_1 and V_2 so that no vertex in V_i is adjacent to any other vertex in V_i for $i = 1$ or 2.

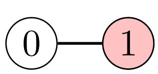
One way to show that a graph is bipartite is to "color" the vertices two different colors, so that no two vertices of the same color are adjacent.

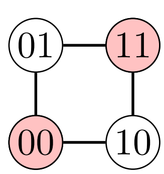
Hypercubes. Let Q_n be the graph with vertex set

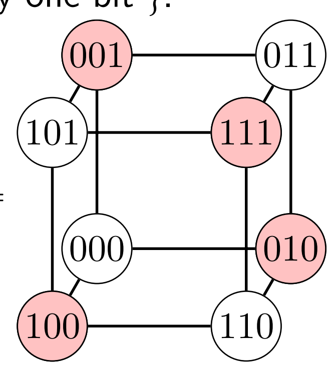
$$V = \{ \text{ bit strings (1's and 0's) of length } n \text{ } \}$$

and edge set

$$E = \{u{-}v \mid u \text{ and } v \text{ differ in exactly one bit } \}.$$

$Q_1 =$ 

$Q_2 =$ 

$Q_3 =$ 

Color vertices with an even number of 0's red.

Graph invariants

To prove that two graphs are isomorphic, you need to find an isomorphism. To show that they're not isomorphic, you have to show that no isomorphism exists, which can be harder! So we look for properties of the graphs that are preserved by isomorphisms. These are called (graph) invariants.

Example: The number of vertices in a graph is an invariant. (If G is isomorphic to H, then there is a bijection between their vertex sets, so those vertex sets must have the same size. Conversely, if G and H have a different number of vertices, then no such bijection exits.)

For example, C_5 and C_6 are different isomorphism classes.

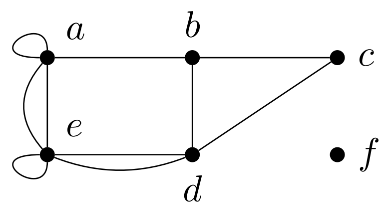
Similarly, the number of edges in a graph is an invariant.

For example, C_5 and K_5 are different isomorphism classes.

Graph invariants

To prove that two graphs are isomorphic, you need to find an isomorphism. To show that they're not isomorphic, you have to show that no isomorphism exists, which can be harder! So we look for properties of the graphs that are preserved by isomorphisms. These are called (graph) invariants.

Example: The degree sequence of a graph is the list of degrees of vertices in the graph, given in decreasing order. For example, the degree sequence of

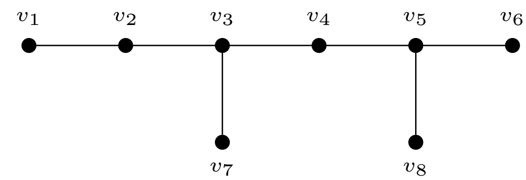
$G =$ 

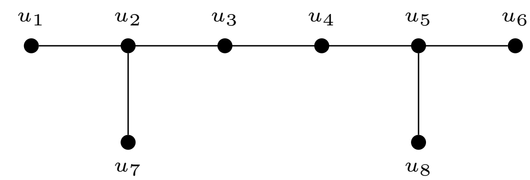
is 6, 5, 4, 3, 2, 0.

(Again, if the degree sequences of G and H differ, then $G \not\cong H$. But if the degree sequences match, the *might* be isomorphic, but they *might not be*.)

Graph invariants

For example, consider the graphs

$G =$ 

$H =$ 

Both of these graphs have the degree sequence $3, 3, 2, 2, 1, 1, 1, 1$. But in G, there's a vertex of degree 1 adjacent to a vertex of degree 2, where as no vertex of degree 1 is adjacent to a vertex of degree 2 in H. So $G \not\cong H$.