Math 365 – Monday 3/25/19

Recall:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \qquad \frac{1-x^n}{1-x} = \sum_{k=0}^{n-1} x^k,$$

$$e^x = \sum_{k=0}^{\infty} x^k/k!, \qquad \text{and} \qquad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

Also, for series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{and} \qquad g(x) = \sum_{k=0}^{\infty} b_k x^k,$$

we have

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \quad \text{and} \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k. \qquad (*)$$

Exercise 37. Recall that the generating function for a sequence $\{a_0, a_1, a_2, \dots,\}$ is

$$\sum_{k=0}^{\infty} a_k x^k.$$

(Considered "formally", i.e. without consideration of convergence.)

Give the generating functions, in terms of their series and in their simplified (closed) form, for each of the following sequences. In many cases, you'll use your answers from Exercise 36 (from last time).

(a) $a_n = n + 1$ for $n = 0, 1, 2, \dots$.

(b) $a_n = 3^n$, for $n = 0, 1, \dots$.

(c) $a_n = \begin{cases} 1 & n = 3k \text{ for some } k \in \mathbb{Z}, \\ 0, & n \neq 3k \text{ for all } k \in \mathbb{Z}, \end{cases}$ for $n = 0, 1, \dots$.

(i.e. a_n is 1 if n is a multiple of 3, and is 0 otherwise).

(d) $a_n = 1/n$ for $n = 1, 2, \dots$.

Exercise 38. For each of the following recursively defined sequences,

(i) solve using generating functions;

(ii) solve using the methods of Section 8.2 and compare to part (i);

(iii) check your answers by comparing the values you get by using your formula and by computing recursively for the first three terms of the sequence.

(a) $a_n = 2a_{n-1} + 3$, $a_0 = 1$.

(b) $a_n = 3a_{n-1} + 4^{n-1}$.

Exercise 39.

(a) Consider the integer partitions of 4.

  (i) Write a corresponding generating function, wherein the coefficient of x^4 is the number of integer partitions of 4.

  (ii) Make explicit the correspondence between the monomials in the expansion of your generating function and the integer partitions of 4 (like we did for the partitions of 5 in class).

(b) Describe a combinatorial problem that is solved by calculating the coefficient of x^6 for the following generating functions.

  (i) $(1 + x + x^2 + x^3)^4$

  (ii) $(1 + x + x^2 + \cdots +)(1 + x^2 + x^4 + x^6)(1 + x^3 + x^6)$

  (iii) $(x + x^2 + x^3)^5$

(c) For each of the following questions, give the corresponding generating function and determine k such that the coefficient of x^k is the answer the question.

  (i) How many different ways can 12 identical action figures can be given to five children so that each child receives at most three action figures?

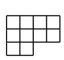
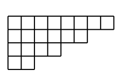
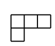
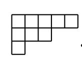
  (ii) How many different ways 10 identical balloons can be given to four children if each child receives at least two balloons?

  (iii) How many different ways are there to choose a dozen bagels from three varieties–plain, onion, and raisin–if at least two bagels of each kind but no more than three plain bagels are chosen?

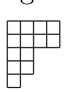
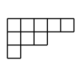
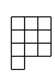
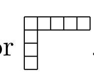
  (iv) How many ways can you make change for \$100 using \$1 bills, \$5 bills, \$10 bills, and \$20 bills?

Exercise 40. (Generating functions for partitions)

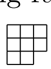
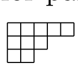
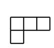
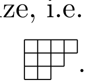
(a) Write the generating function for partitions with even-sized parts, i.e.

 or  but not  or .

(b) Write the generating function for partitions with no more than two parts of each size, i.e.

 or  but not  or .

(c) Write the generating function for partitions parts all of prime size, i.e.

 or  but not  or .

(The prime numbers are the integers p greater than 1 that are divisible by 1 and p but no other positive integers.)

Taylor series to know and love:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + x^n \qquad \text{(finite)}$$

$$\frac{1-x^n}{1-x} = \sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} \qquad \text{(finite)}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \qquad \text{(infinite)}$$

$$e^x = \sum_{k=0}^{\infty} x^k/k! = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \qquad \text{(infinite)}$$

The lefthand side of each is called the closed form for the series.

New Series from old: Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$.

Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k)x^k \quad \text{and} \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k.$$

You can also differentiate and integrate series to get new series.

Section 8.4: Generating functions.

A generating function for a sequence $\{a_k\}_{k=0,1,\ldots}$ is the series

$$\sum_{k=0}^{\infty} a_k x^k.$$

("Formal": forget about convergence!)

When possible, we rewrite the generating function in terms of a simple expression of elementary functions, which we call closed solutions.

For example, the generating function for the sequence $1, 1, 1, \cdots = \{1\}_{k=0,1,\ldots}$ is

$$\sum_{k=0}^{\infty} 1 * x^k = \frac{1}{1-x}.$$

The generating function for the sequence $1, \frac{1}{2}, \frac{1}{6}, \cdots = \{1/n!\}$ is

$$\sum_{n=0}^{\infty} x^n/n! = e^x.$$

Section 8.4: Generating functions.

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Not every generating function has a nice closed form, but that shouldn't stop you from writing it down. For example, the generating function for the sequence $1, 0, 2, 0, 3, 0, \ldots$ is

$$1 + 0 * x + 2x^2 + 0 * x^3 + 3x^4 + \cdots = \sum_{k=0}^{\infty} (k+1) * x^{2k}.$$

And the generating function for the sequence$0, 0, 2^2, 3^2, 0, 5^2, 0, 7^2, \ldots$, i.e. $a_n = n^2$ if n is prime and $a_n = 0$ otherwise is

$$2^2x^2 + 3^2x^3 + 5^2x^5 + 7^2x^2 + \cdots = \sum_{p \text{ prime}} p^2 x^p.$$

A generating function for a sequence $\{a_k\}_{k=0,1,\ldots}$ is the series

$$\sum_{k=0}^{\infty} a_k x^k.$$

("Formal": forget about convergence!)

When possible, we rewrite the generating function in terms of a simple expression of elementary functions, which we call closed solutions.

Note that a finite sequence $a_0, a_1, \ldots, a_n$ is the same as the infinite sequence $a_0, a_1, \ldots, a_n, 0, 0, \ldots$; similarly, the generating function for a finite sequence will be a finite degree polynomial:

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + 0 + 0 + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

For example, for a fixed n, the generating function for the sequence $\{\binom{n}{k}\}_{k=0,1,\ldots,n}$ is

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (x+1)^n.$$

You try Exercise 37

First application: solving recurrence relations

Take a generating function for some sequence $\{a_n\}$:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Notice that

$$xG(x) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$x^2 G(x) = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$\vdots$$

$$x^d G(x) = a_0 x^d + a_1 x^{d+1} + a_2 x^{d+2} + \cdots = \sum_{n=d}^{\infty} a_{n-d} x^n.$$

(Rewrite sums so that the power of x matches the index, to make it easier to collect "like terms" when adding series!)

First application: solving recurrence relations

So say I have a sequence $\{a_n\}$ that satisfies the recurrence relation $a_n = 3a_{n-1}$. (Sanity check: we already know the general solution should look like $\boxed{a_n = a_0 3^n}$.) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$
\begin{aligned}
G(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\
&= a_0 + x(a_1 + a_2 x + \cdots) && \text{Set aside } d \text{ terms,} \\
& && \text{(where } d = \text{degree of recurrence)} \\
&= a_0 + x \sum_{n=0}^{\infty} a_{n+1} x^n && \text{and factor out } x^d \text{ from the rest.} \\
&= a_0 + x \sum_{n=0}^{\infty} 3a_n x^n && \text{Plug in the recurrence relation.} \\
&= a_0 + 3x \sum_{n=0}^{\infty} a_n x^n && \text{Simplify.} \\
&= a_0 + 3xG(x). && \text{Return to closed form.}
\end{aligned}
$$

$$\boxed{G(x) = a_0 + 3xG(x).}$$

First application: solving recurrence relations

So say I have a sequence $\{a_n\}$ that satisfies the recurrence relation $a_n = 3a_{n-1}$. (Sanity check: we already know the general solution should look like $\boxed{a_n = a_0 3^n}$.) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\boxed{G(x) = a_0 + 3xG(x).}$$

Now solve for $G(x)$:

$$a_0 = G(x) - 3xG(x) = (1-3x)G(x);$$

and so for $x \neq 1/3$,

$$
\begin{aligned}
G(x) = \frac{a_0}{1-3x} &= a_0 \left(\frac{1}{1-y} \right)\bigg|_{y=3x} \\
&= a_0 \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} (a_0 3^n) x^n.
\end{aligned}
$$

Now compare to the original formula for $G(x)$! This shows that $a_n = a_0 3^n$ (as expected).

Ex 2: suppose I have a sequence satisfying

$$a_n = 9a_{n-2} + 10^{n-2} \text{ with } a_0 = 3 \text{ and } a_1 = 2.$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x + x^2(a_2 + a_3 x + \cdots)$$

Set aside d terms, (where d = degree of recurrence)

$$= a_0 + a_1 x + x^2 \sum_{n=0}^{\infty} a_{n+2} x^n$$

and factor out x^d from the rest.

$$= a_0 + a_1 x + x^2 \sum_{n=0}^{\infty} (9a_n + 10^n) x^n$$

Plug in the recurrence relation.

$$= a_0 + a_1 x + 9x^2 \sum_{n=0}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} (10x)^n$$

Expand and simplify.

$$= a_0 + a_1 x + 9x^2 G(x) + x^2 \left(\frac{1}{1-10x}\right).$$

Return to closed forms.

Ex 2: suppose I have a sequence satisfying

$$a_n = 9a_{n-2} + 10^{n-2} \text{ with } a_0 = 3 \text{ and } a_1 = 2.$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\boxed{G(x) = a_0 + a_1 x + 9x^2 G(x) + x^2 \left(\frac{1}{1-10x}\right)}$$

Now solve for $G(x)$:

$$a_0 + a_1 x + x^2 \left(\frac{1}{1-10x}\right) = G(x) - 9x^2 G(x) = (1-9x^2)G(x);$$

So

$$G(x) = \frac{(a_0 + a_1 x)(1-10x) + x^2}{(1-10x)(1-9x^2)} = \frac{a_0 + (a_1 - 10a_0)x + (1-10a_1)x^2}{(1-10x)(1+3x)(1-3x)}$$

$$= \frac{3 - 28x - 19x^2}{(1-10x)(1+3x)(1-3x)}$$

$$= \frac{1}{1-10x} + \left(\frac{46}{39}\right)\frac{1}{1-(-3x)} + \left(\frac{1}{91}\right)\frac{1}{1-10x}$$

Review partial fractions decomposition!

Ex 2: suppose I have a sequence satisfying

$$a_n = 9a_{n-2} + 10^{n-2} \text{ with } a_0 = 3 \text{ and } a_1 = 2.$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\boxed{G(x) = a_0 + a_1 x + 9x^2 G(x) + x^2 \left(\frac{1}{1-10x}\right)}$$

Now solve for $G(x)$:

$$G(x) = \frac{1}{1-10x} + \left(\frac{46}{39}\right) \frac{1}{1-(-3x)} + \left(\frac{1}{91}\right) \frac{1}{1-10x}$$

Review partial fractions decomposition!

Putting back into series form, we get

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} 10^n x^n + \left(\frac{46}{39}\right) \sum_{n=0}^{\infty} (-3)^n x^n + \left(\frac{1}{91}\right) \sum_{n=0}^{\infty} 3^n x^n \\ &= \sum_{n=0}^{\infty} \left(10^n + \left(\frac{46}{39}\right)(-3)^n + \left(\frac{1}{91}\right) 3^n\right) x^n. \end{aligned}$$

So

$$a_n = 10^n + \left(\frac{46}{39}\right)(-3)^n + \left(\frac{1}{91}\right) 3^n$$

Try Ex 38

Counting problems and Generating functions

Example: What is the coefficient on x^{12} in

$$\underbrace{(x^2 + x^3 + x^4 + x^5)}_{e_1, \text{ glazed}} \underbrace{(x^4 + x^5)}_{e_2, \text{ choc.}} \underbrace{(x^1 + x^2 + x^3)}_{e_3, \text{ jelly}}?$$

This is equivalent to the question "How many integer solutions are there to the equation

$$e_1 + e_2 + e_3 = 12$$

with

$$2 \leqslant e_1 \leqslant 5, \qquad 4 \leqslant e_2 \leqslant 5, \qquad 1 \leqslant e_3 \leqslant 3?"$$

Which is the same as "How many ways can you pick 12 doughnuts to bring to the office if you've had requests for at least 2 glazed, 4 chocolate, and one jelly-filled, but when you get to the store, they only have 5 glazed, 5 chocolate, and 3 jelly-filled left?"

Example: Use a generating function to answer the question "How many non-negative integer solutions are there to

$$e_1 + e_2 + e_3 = 10$$

where e_2 is a multiple of 2 and e_3 is a multiple of 3?"

The answer is the same as the coefficient of x^{10} in

$$\underbrace{(1 + x + x^2 + \cdots)}_{e_1}\underbrace{(1 + x^2 + x^4 + x^6 + \cdots)}_{e_2}\underbrace{(1 + x^3 + x^6 + x^9 + \cdots)}_{e_3},$$

which is the same as the coefficient of x^{10} in

$$\underbrace{(1 + x + x^2 + \cdots + x^{10})}_{e_1}\underbrace{(1 + x^2 + x^4 + \cdots + x^{10})}_{e_2}\underbrace{(1 + x^3 + x^6 + x^9)}_{e_3},$$

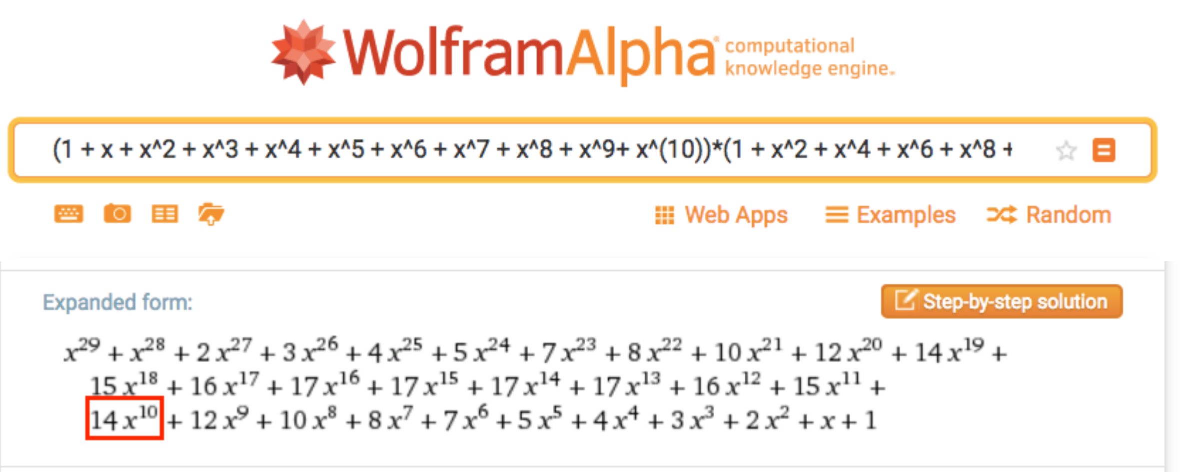
since we would never use any terms that came from x^a for $a > 10$. This is something we can plug into a calculator like WolframAlpha.

Example: Use a generating function to answer the question "How many non-negative integer solutions are there to

$$e_1 + e_2 + e_3 = 10$$

where e_2 is a multiple of 2 and e_3 is a multiple of 3?"

The answer is the same as the coefficient of x^{10} in...
This is something we can plug into a calculator like WolframAlpha:



Integer partitions

How many integer partitions are there of 5?

This is the same as the coefficient of x^5 in

$$(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5)$$

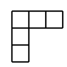
$$= \big((x^1)^0+(x^1)^1+(x^1)^2+(x^1)^3+(x^1)^4+(x^1)^5\big) \quad \text{(pts of length 1)}$$

$$\big((x^2)^0+(x^2)^1+(x^2)^2\big) \quad \text{(pts of length 2)}$$

$$\big((x^3)^0+(x^3)^1\big) \quad \text{(pts of length 3)}$$

$$\big((x^4)^0+(x^4)^1\big) \quad \text{(pts of length 4)}$$

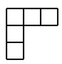
$$\big((x^5)^0+(x^5)^1\big) \quad \text{(pts of length 5)}$$

Why? For example, consider the partition .

Integer partitions

Counting integer partitions of 5 by looking at the coeff. of x^5 in

$$(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5)\dots$$

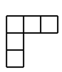
 corresponds to

$(x^1)^2$ from first factor, since there are 2 parts of length 1,
$1=(x^2)^0$ from second factor, since there are 0 parts of length 2,
$(x^3)^1$ from third factor, since there is 1 part of length 3,
$1=(x^4)^0$ from fourth factor, since there are 0 parts of length 4, and
$1=(x^5)^0$ from fourth factor, since there are 1 parts of length 5.

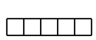
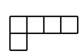
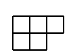
Integer partitions

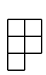
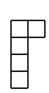
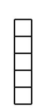
Counting integer partitions of 5 by looking at the coeff. of x^5 in

$$(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5)\dots$$

 corresponds to $(x^1)^2 * 1 * (x^3)^1 * 1 * 1$.

Similarly, the correspondence between the other partitions of 5 and the monomials goes like

 $1*1*1*1*(x^5)^1$	 $x*1*1*(x^4)^1*1$	 $1*(x^2)^1*(x^3)^1*1*1$

 $(x^1)^1*(x^2)^2*1*1*1$	 $(x^1)^3*(x^2)^1*1*1*1$	 $(x^1)^5*1*1*1*1$

Notice that the coefficient of x^5 in the polynomial from the previous slide is the same as the coefficient of x^5 in

$$\left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{i=0}^{\infty} x^{3i}\right)\left(\sum_{i=0}^{\infty} x^{4i}\right)\left(\sum_{i=0}^{\infty} x^{5i}\right)$$

$$= \prod_{k=1}^{5}\left(\sum_{i=0}^{\infty} x^{ki}\right) = \prod_{k=1}^{5}\left(\frac{1}{1-x^k}\right).$$

Which is the same as the coefficient of x^5 in

$$\left(\sum_{i=0}^{\infty} x^i\right) * \left(\sum_{i=0}^{\infty} x^{2i}\right) * \left(\sum_{i=0}^{\infty} x^{3i}\right) * \left(\sum_{i=0}^{\infty} x^{4i}\right) * \left(\sum_{i=0}^{\infty} x^{5i}\right) * \underbrace{\left(\sum_{i=0}^{\infty} x^{6i}\right)}_{\substack{\text{must use} \\ \text{the 1 term}}} * \underbrace{\left(\sum_{i=0}^{\infty} x^{7i}\right)}_{\substack{\text{must use} \\ \text{the 1 term}}} \cdots$$

Notice that the coefficient of x^5 in the polynomial from the previous slide is the same as the coefficient of x^5 in

$$\left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{i=0}^{\infty} x^{3i}\right)\left(\sum_{i=0}^{\infty} x^{4i}\right)\left(\sum_{i=0}^{\infty} x^{5i}\right)$$

$$= \prod_{k=1}^{5}\left(\sum_{i=0}^{\infty} x^{ki}\right) = \prod_{k=1}^{5}\left(\frac{1}{1-x^k}\right).$$

Which is the same as the coefficient of x^5 in

$$\left(\sum_{i=0}^{\infty} x^i\right) * \left(\sum_{i=0}^{\infty} x^{2i}\right) * \left(\sum_{i=0}^{\infty} x^{3i}\right) * \cdots = \prod_{k=1}^{\infty}\left(\sum_{i=0}^{\infty} x^{ki}\right)$$

So in general, the number of integer partitions of n, denoted $p(n)$, is the coefficient of x^n in

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}\left(\frac{1}{1-x^k}\right).$$