Taylor series to know and love:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + x^n \qquad \text{(finite)}$$

$$\frac{1-x^n}{1-x} = \sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} \qquad \text{(finite)}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \qquad \text{(infinite)}$$

$$e^x = \sum_{k=0}^{\infty} x^k/k! = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \qquad \text{(infinite)}$$

The lefthand side of each is called the closed form for the series.

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New Series from old: Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$.

Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \quad \text{and} \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k.$$

You can also differentiate and integrate series to get new series.

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A generating function for a sequence $\{a_k\}_{k=0,1,\ldots}$ is the series

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When possible, we rewrite the generating function in terms of a simple expression of elementary functions, which we call closed solutions.

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$$2^2x^2 + 3^2x^3 + 5^2x^5 + 7^2x^2 + \cdots = \sum_{p \text{ prime}} p^2 x^p.$$

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$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + 0 + 0 + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

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You try Exercise 37

First application: solving recurrence relations

Take a generating function for some sequence $\{a_n\}$:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

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$$\begin{aligned} G(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\ &= a_0 + x(a_1 + a_2 x + \cdots) \end{aligned}$$

Set aside d terms,

(where $d =$ degree of recurrence)

and factor out x^d from the rest.

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$$= a_0 + x \sum_{n=0}^{\infty} 3a_n x^n$$

Plug in the recurrence relation.

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G(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\
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&= a_0 + x \sum_{n=0}^{\infty} a_{n+1} x^n && \text{and factor out } x^d \text{ from the rest.} \\
&= a_0 + x \sum_{n=0}^{\infty} 3a_n x^n && \text{Plug in the recurrence relation.} \\
&= a_0 + 3x \sum_{n=0}^{\infty} a_n x^n && \text{Simplify.}
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&= a_0 + 3xG(x). && \text{Return to closed form.}
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$$\begin{aligned} G(x) &= \frac{a_0}{1-3x} = a_0 \left(\frac{1}{1-y} \right) \Big|_{y=3x} \\ &= a_0 \sum_{n=0}^{\infty} (3x)^n \end{aligned}$$

First application: solving recurrence relations

So say I have a sequence $\{a_n\}$ that satisfies the recurrence relation $a_n = 3a_{n-1}$. (Sanity check: we already know the general solution should look like $\boxed{a_n = a_0 3^n}$.) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\boxed{G(x) = a_0 + 3xG(x).}$$

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Now compare to the original formula for $G(x)$! This shows that $a_n = a_0 3^n$ (as expected).

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and factor out x^d from the rest.

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$$= a_0 + a_1 x + x^2 \sum_{n=0}^{\infty} (9a_n + 10^n) x^n$$

Plug in the recurrence relation.

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$$= a_0 + a_1 x + 9x^2 \sum_{n=0}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} (10x)^n$$

Expand and simplify.

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Expand and simplify.

$$= a_0 + a_1 x + 9x^2 G(x) + x^2 \left(\frac{1}{1 - 10x} \right).$$

Return to closed forms.

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$a_n = 9a_{n-2} + 10^{n-2}$ with $a_0 = 3$ and $a_1 = 2$.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

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Now solve for $G(x)$:

$$a_0 + a_1 x + x^2 \left(\frac{1}{1 - 10x} \right) = G(x) - 9x^2 G(x)$$

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Now solve for $G(x)$:

$$a_0 + a_1 x + x^2 \left(\frac{1}{1 - 10x} \right) = G(x) - 9x^2 G(x) = (1 - 9x^2) G(x);$$

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$$G(x) = \frac{(a_0 + a_1 x)(1-10x) + x^2}{(1-10x)(1-9x^2)} = \frac{a_0 + (a_1 - 10a_0)x + (1-10a_1)x^2}{(1-10x)(1+3x)(1-3x)}$$

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$$= \frac{3 - 28x - 19x^2}{(1 - 10x)(1 + 3x)(1 - 3x)}$$

$$= \frac{1}{1 - 10x} + \left(\frac{46}{39} \right) \frac{1}{1 - (-3x)} + \left(\frac{1}{91} \right) \frac{1}{1 - 10x}$$

Review partial fractions decomposition!

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$$G(x) = \sum_{n=0}^{\infty} 10^n x^n + \left(\frac{46}{39}\right) \sum_{n=0}^{\infty} (-3)^n x^n + \left(\frac{1}{91}\right) \sum_{n=0}^{\infty} 3^n x^n$$

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$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} 10^n x^n + \left(\frac{46}{39}\right) \sum_{n=0}^{\infty} (-3)^n x^n + \left(\frac{1}{91}\right) \sum_{n=0}^{\infty} 3^n x^n \\ &= \sum_{n=0}^{\infty} \left(10^n + \left(\frac{46}{39}\right)(-3)^n + \left(\frac{1}{91}\right) 3^n \right) x^n. \end{aligned}$$

So

$$a_n = 10^n + \left(\frac{46}{39}\right)(-3)^n + \left(\frac{1}{91}\right) 3^n$$

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Try Ex 38

Counting problems and Generating functions

Example: What is the coefficient on x^{12} in

$$(x^2 + x^3 + x^4 + x^5)(x^4 + x^5)(x^1 + x^2 + x^3)?$$

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$$\underbrace{(x^2 + x^3 + x^4 + x^5)}_{e_1}\underbrace{(x^4 + x^5)}_{e_2}\underbrace{(x^1 + x^2 + x^3)}_{e_3}?$$

This is equivalent to the question "How many integer solutions are there to the equation

$$e_1 + e_2 + e_3 = 12$$

with

$$2 \leqslant e_1 \leqslant 5, \qquad 4 \leqslant e_2 \leqslant 5, \qquad 1 \leqslant e_3 \leqslant 3?''$$

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Which is the same as "How many ways can you pick 12 doughnuts to bring to the office if you've had requests for at least 2 glazed, 4 chocolate, and one jelly-filled, but when you get to the store, they only have 5 glazed, 5 chocolate, and 3 jelly-filled left?"

Example: Use a generating function to answer the question "How many non-negative integer solutions are there to

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where e_2 is a multiple of 2 and e_3 is a multiple of 3?"

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The answer is the same as the coefficient of x^{10} in

$$\underbrace{(1 + x + x^2 + \cdots)}_{e_1}\underbrace{(1 + x^2 + x^4 + x^6 + \cdots)}_{e_2}\underbrace{(1 + x^3 + x^6 + x^9 + \cdots)}_{e_3}$$

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since we would never use any terms that came from x^a for $a > 10$.

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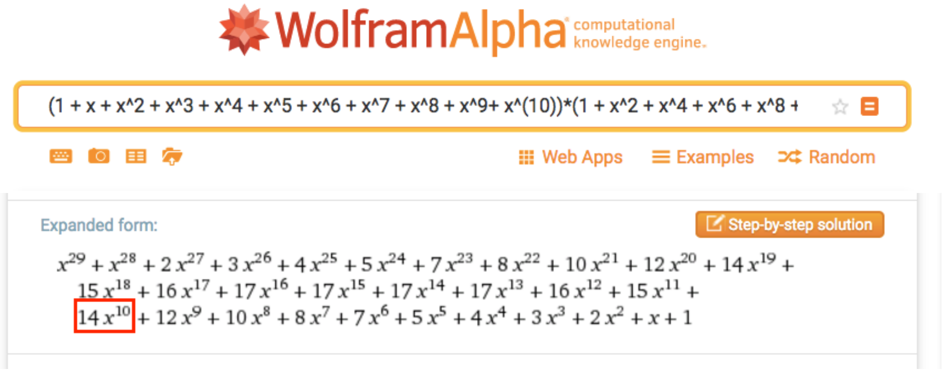

Integer partitions

How many integer partitions are there of 5?

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$$
\begin{aligned}
&(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5)\\
&\quad = \big((x^1)^0+(x^1)^1+(x^1)^2+(x^1)^3+(x^1)^4+(x^1)^5\big)\\
&\qquad\quad \big((x^2)^0+(x^2)^1+(x^2)^2\big)\\
&\qquad\quad \big((x^3)^0+(x^3)^1\big)\\
&\qquad\quad \big((x^4)^0+(x^4)^1\big)\\
&\qquad\quad \big((x^5)^0+(x^5)^1\big)
\end{aligned}
$$

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$$(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5)$$
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$$\big((x^3)^0+(x^3)^1\big)$$
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$$\big((x^5)^0+(x^5)^1\big)$$

Why?

Integer partitions

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$$
\begin{aligned}
&(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5) \\
&\quad = \big((x^1)^0+(x^1)^1+(x^1)^2+(x^1)^3+(x^1)^4+(x^1)^5\big) \quad \text{(pts of length 1)} \\
&\qquad \big((x^2)^0+(x^2)^1+(x^2)^2\big) \quad \text{(pts of length 2)} \\
&\qquad \big((x^3)^0+(x^3)^1\big) \quad \text{(pts of length 3)} \\
&\qquad \big((x^4)^0+(x^4)^1\big) \quad \text{(pts of length 4)} \\
&\qquad \big((x^5)^0+(x^5)^1\big) \quad \text{(pts of length 5)}
\end{aligned}
$$

Why?

Integer partitions

How many integer partitions are there of 5?

This is the same as the coefficient of x^5 in

$$
\begin{aligned}
&(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5) \\
&\quad = \big((x^1)^0+(x^1)^1+(x^1)^2+(x^1)^3+(x^1)^4+(x^1)^5\big) \quad \text{(pts of length 1)} \\
&\qquad \big((x^2)^0+(x^2)^1+(x^2)^2\big) \qquad \text{(pts of length 2)} \\
&\qquad \big((x^3)^0+(x^3)^1\big) \qquad \text{(pts of length 3)} \\
&\qquad \big((x^4)^0+(x^4)^1\big) \qquad \text{(pts of length 4)} \\
&\qquad \big((x^5)^0+(x^5)^1\big) \qquad \text{(pts of length 5)}
\end{aligned}
$$

Why? For example, consider the partition .

Integer partitions

Counting integer partitions of 5 by looking at the coeff. of x^5 in

$$(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5)\dots$$

 corresponds to

$(x^1)^2$ from first factor, since there are 2 parts of length 1,
$1=(x^2)^0$ from second factor, since there are 0 parts of length 2,
$(x^3)^1$ from third factor, since there is 1 part of length 3,
$1=(x^4)^0$ from fourth factor, since there are 0 parts of length 4, and
$1=(x^5)^0$ from fourth factor, since there are 1 parts of length 5.

Integer partitions

Counting integer partitions of 5 by looking at the coeff. of x^5 in

$$(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5)\dots$$

 corresponds to $(x^1)^2 * 1 * (x^3)^1 * 1 * 1$.

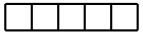
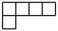
Integer partitions

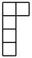
Counting integer partitions of 5 by looking at the coeff. of x^5 in

$$(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5)\dots$$

 corresponds to $(x^1)^2 * 1 * (x^3)^1 * 1 * 1$.

Similarly, the correspondence between the other partitions of 5 and the monomials goes like

		
$1 * 1 * 1 * 1 * (x^5)^1$	$x * 1 * 1 * (x^4)^1 * 1$	$1 * (x^2)^1 * (x^3)^1 * 1 * 1$

		
$(x^1)^1 * (x^2)^2 * 1 * 1 * 1$	$(x^1)^3 * (x^2)^1 * 1 * 1 * 1$	$(x^1)^5 * 1 * 1 * 1 * 1$

Notice that the coefficient of x^5 in the polynomial from the previous slide is the same as the coefficient of x^5 in

$$\left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{i=0}^{\infty} x^{3i}\right)\left(\sum_{i=0}^{\infty} x^{4i}\right)\left(\sum_{i=0}^{\infty} x^{5i}\right)$$

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$$= \prod_{k=1}^{5}\left(\sum_{i=0}^{\infty} x^{ki}\right)$$

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$$= \prod_{k=1}^{5}\left(\sum_{i=0}^{\infty} x^{ki}\right) = \prod_{k=1}^{5}\left(\frac{1}{1-x^k}\right).$$

Which is the same as the coefficient of x^5 in

$$\left(\sum_{i=0}^{\infty} x^i\right) * \left(\sum_{i=0}^{\infty} x^{2i}\right) * \left(\sum_{i=0}^{\infty} x^{3i}\right) * \left(\sum_{i=0}^{\infty} x^{4i}\right) * \left(\sum_{i=0}^{\infty} x^{5i}\right) * \underbrace{\left(\sum_{i=0}^{\infty} x^{6i}\right)}_{\text{must use the 1 term}} * \underbrace{\left(\sum_{i=0}^{\infty} x^{7i}\right)}_{\text{must use the 1 term}} \cdots$$

Notice that the coefficient of x^5 in the polynomial from the previous slide is the same as the coefficient of x^5 in

$$\left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{i=0}^{\infty} x^{2i}\right)\left(\sum_{i=0}^{\infty} x^{3i}\right)\left(\sum_{i=0}^{\infty} x^{4i}\right)\left(\sum_{i=0}^{\infty} x^{5i}\right)$$

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Which is the same as the coefficient of x^5 in

$$\left(\sum_{i=0}^{\infty} x^i\right) * \left(\sum_{i=0}^{\infty} x^{2i}\right) * \left(\sum_{i=0}^{\infty} x^{3i}\right) * \cdots = \prod_{k=1}^{\infty}\left(\sum_{i=0}^{\infty} x^{ki}\right)$$

So in general, the number of integer partitions of n, denoted $p(n)$, is the coefficient of x^n in

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}\left(\frac{1}{1-x^k}\right).$$