Math 365 – Wednesday 3/20/19 – 8.2 and 8.4

Exercise 35. Consider the recurrence relation

$$a_n = 8a_{n-2} - 16a_{n-4} + F(n).$$

(a) Write the associated homogeneous recursion relation and solve for its general solution $\{h_n\}$.

(b) For each of the following $F(n)$, write the general form for the particular solution (don't solve for the unknowns).

(i) $F(n) = n^3$
(ii) $F(n) = n2^n$
(iii) $F(n) = (n^2 - 2)(-2)^n$
(iv) $F(n) = 2$
(v) $F(n) = (-2)^n$
(vi) $F(n) = n^2 4^n$
(vii) $F(n) = n^4 2^n$

(c) Find the general solution to

$$a_n = 8a_{n-2} - 16a_{n-4} + (-2)^n.$$

(d) Find the general solution to

$$a_n = 8a_{n-2} - 16a_{n-4} + n^3.$$

(e) Pick an example of appropriate initial conditions for the sequence in part (c), and solve for the corresponding specific solution. Check your answer by computing the first 6 terms of the sequence both recursively and using your closed formula.

Exercise 36.

(a) Explicitly compute the series for

$$\frac{1}{(1-x)^3} \quad \text{and} \quad \frac{1}{(1-x)^4}$$

by taking derivatives and rescaling appropriately. Conjecture what the general formula for the series for $1/(1-x)^n$.

(b) Substitute $y = 3x$ into the series for $\frac{1}{1-y}$ to get the series for $\dfrac{1}{1-3x}$. What is the series for $1/(1-4x)$? What is the series for $1/(1+x)$?

(c) Substitute $y = x^3$ into the series for $\dfrac{1}{1-y}$ to get the series for $\dfrac{1}{1-x^3}$. What is the series for $1/(1-x^4)$? What is the series for $1/(1-2x^3)$?

(d) Use the fact that $\dfrac{d}{dx}(\ln(1-x)) = -\dfrac{1}{1-x}$ to compute the series for $\ln(1-x)$. (Integrate, and pick your "$+C$" to make it so that evaluating your series at $x = 0$ matches correctly with evaluating $\ln(1-x)$.)

Today: Section 8.2, continued.

Last time:
To solve recurrence relations, the best we can do is make educated guesses, according to what kind of relation we have.

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Solutions to this kind of relation come in the form r^n, where r is a root of the characteristic equation, which is obtained by plugging in r^n, dividing through by r^{n-k}, and solving for 0:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0.$$

The roots of this equation are called the characteristic roots.

Example from last time: plugging $a_n = r^n$ into the recursion relation

$$a_n = -a_{n-1} + a_{n-2} + a_{n-3}$$

gets

$$r^n = -r^{n-1} + r^{n-2} + r^{n-3},$$

which is true if and only if

$$\begin{aligned} 0 &= r^n + r^{n-1} - r^{n-2} - r^{n-3} \\ &= r^{n-3}(r^3 + r^2 - r - 1) \end{aligned}$$

which is true if and only if $r = 0$ or

$$\underbrace{0 = r^3 + r^2 - r - 1}_{\text{characteristic equation}} = (r+1)^2(r-1).$$

The characteristic roots are $r_1 = 1$ (with multiplicity 1) and $r_2 = -1$ (with multiplicity 2).

Theorem: Solving linear homogeneous recurrences

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation has roots $r_1, r_2, \ldots, r_\ell$ with multiplicities $m_1, m_2, \ldots, m_\ell$. Then a sequence $\{a_n\}_{n \in \mathbb{N}}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$a_n = p_1(n) r_1^n + p_2(n) r_2^n + \cdots + p_\ell(n) r_\ell^n,$$

where $p_i(n)$ are polynomials in n of degree $m_i - 1$.

Example: $a_n = a_{n-1} + 5a_{n-2} - a_{n-3} - 8a_{n-4} - 4a_{n-5}$.

Characteristic equation:

$$0 = r^5 - r^4 - 5r^3 + r^2 + 8r + 4 = (r+1)^3 (r-2)^2.$$

General solution:

$$a_n = (\alpha_0 + \alpha_1 n + \alpha_2 n^2)(-1)^n + (\beta_0 + \beta_1 n)(2)^n.$$

Non-homogeneous equations

Suppose your recurrence is linear and constant coefficient in a_i's, but is not homogeneous. In other words, it is in the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

where $F(n)$ is a function only in n (no a_i's). The relation

$$h_n = c_1 h_{n-1} + c_2 h_{n-2} + \cdots + c_k h_{n-k} \quad \text{(so that } a_n = h_n + F(n)\text{)}$$

is called the associated homogeneous recurrence relation.

Ex: $a_n = 3a_{n-1} + 2n$.

$F(n) = 2n$, Assoc. hom: $h_n = 3h_{n-1}$.

Ex: $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

$F(n) = 7^n$, Assoc. hom: $h_n = 5h_{n-1} - 6h_{n-2}$.

The following theorem says that if we can find *one solution* to a_n, then the general solutions to h_n will help us build all the rest of the solutions to a_n.

Theorem: Solving non-homogeneous equations

(a) If $\{\hat{a}_n\}_{n\in\mathbb{N}}$ is one solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n = \hat{a}_n + h_n\}_{n\in\mathbb{N}}$, where $\{h_n\}_{n\in\mathbb{N}}$ is a solution of the associated homogeneous recurrence relation.

(b) Finding $\hat{a}_n$: If $F(n) = Q(n)R^n$, where

- $Q(n)$ is a polynomial in n, and
- R is a constant,

then there is a solution to a_n of the form

$$\hat{a}_n = n^m q(n) R^n$$

where

- $\deg(q(n)) \leqslant \deg(Q(n))$, and
- $m =$ mult. of R in the characteristic equation (possibly 0).

Ex: Find all solutions to $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

1. Break the sequence into two parts: homogeneous h_n and a function of n: $\boxed{a_n = h_n + F(n)}$.
2. Solve for h_n:
   - compute the characteristic equation;
   - factor to compute roots and multiplicities;
   - build the general solution to h_n.
3. Fine one solution $\hat{a}_n$ by guessing something of a similar form.

   If $F(n) = Q(n)R^n$, guess $\hat{a}_n = n^m q(n) R^n$

   where $m =$ mult of R, and $q_n = b_0 + b_1 n + b_2 n^2 + \cdots + b_d n^d$ where $d = \deg(Q(n))$.

Example: $a_n = 4a_{n-1} + 3a_{n-2} - 18a_{n-3} + 4^n$

Homog: $h_n = 4h_{n-1} + 3h_{n-2} - 18h_{n-3}$ and $F(n) = 4^n$.

Char eq: $0 = r^3 - 4r^2 - 3r + 18 = (r+2)(r-3)^2$

Homog sol: $h_n = \alpha(-2)^n + (\alpha + \beta n)3^n$

Particular solution guess: $\hat{a}_n = b4^n$ (gives $b = 32/3$)

General sol: $a_n = \alpha(-2)^n + (\alpha + \beta n)3^n + \frac{32}{3}4^n$.

1. Break the sequence into two parts: homogeneous h_n and a function of n: $\boxed{a_n = h_n + F(n)}$.
2. Solve for h_n:
   - compute the characteristic equation;
   - factor to compute roots and multiplicities;
   - build the general solution to h_n.
3. Fine one solution $\hat{a}_n$ by guessing something of a similar form.

   If $F(n) = Q(n)R^n$, guess $\hat{a}_n = n^m q(n) R^n$

   where $m =$ mult of R, and $q_n = b_0 + b_1 n + b_2 n^2 + \cdots + b_d n^d$ where $d = \deg(Q(n))$.

Example: $a_n = 4a_{n-1} + 3a_{n-2} - 18a_{n-3} + n4^n$

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Homog sol: $h_n = \alpha(-2)^n + (\alpha + \beta n)3^n$

Particular solution guess: $\hat{a}_n = (b_0 + b_1 n)4^n$

(gives $b_0 = 1376/9$ and $b_1 = -63/3$)

General sol: $a_n = \alpha(-2)^n + (\alpha + \beta n)3^n + (\frac{1376}{9} - \frac{63}{3}n)4^n$.

1. Break the sequence into two parts: homogeneous h_n and a function of n: $\boxed{a_n = h_n + F(n)}$.
2. Solve for h_n:
   - compute the characteristic equation;
   - factor to compute roots and multiplicities;
   - build the general solution to h_n.
3. Fine one solution $\hat{a}_n$ by guessing something of a similar form.

   If $F(n) = Q(n)R^n$, guess $\hat{a}_n = n^m q(n) R^n$

   where $m =$ mult of R, and $q_n = b_0 + b_1 n + b_2 n^2 + \cdots + b_d n^d$ where $d = \deg(Q(n))$.

Example: $a_n = 4a_{n-1} + 3a_{n-2} - 18a_{n-3} + 3^n$

Homog: $h_n = 4h_{n-1} + 3h_{n-2} - 18h_{n-3}$ and $F(n) = 3^n$.

Char eq: $0 = r^3 - 4r^2 - 3r + 18 = (r+2)(r-3)^2$

Homog sol: $h_n = \alpha(-2)^n + (\alpha + \beta n)3^n$

Particular solution guess: $\hat{a}_n = bn^2 3^n$

(gives $b = 3/2$)

General sol: $a_n = \alpha(-2)^n + (\alpha + \beta n)3^n + \frac{3}{2}n^2 3^n$.

1. Break the sequence into two parts: homogeneous h_n and a function of n: $\boxed{a_n = h_n + F(n)}$.
2. Solve for h_n:
   - compute the characteristic equation;
   - factor to compute roots and multiplicities;
   - build the general solution to h_n.
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   If $F(n) = Q(n)R^n$, guess $\hat{a}_n = n^m q(n) R^n$

   where $m =$ mult of R, and $q_n = b_0 + b_1 n + b_2 n^2 + \cdots + b_d n^d$ where $d = \deg(Q(n))$.

Example: $a_n = 4a_{n-1} + 3a_{n-2} - 18a_{n-3} + n3^n$

Homog: $h_n = 4h_{n-1} + 3h_{n-2} - 18h_{n-3}$ and $F(n) = 3^n$.

Char eq: $0 = r^3 - 4r^2 - 3r + 18 = (r+2)(r-3)^2$

Homog sol: $h_n = \alpha(-2)^n + (\alpha + \beta n)3^n$

Particular solution guess: $\hat{a}_n = (b_0 + b_1 n)n^2 3^n$

(gives $b_0 = 21/50$ and $b_1 = 1/10$)

General sol: $a_n = \alpha(-2)^n + (\alpha + \beta n)3^n + (\frac{21}{50} + \frac{1}{10}n)n^2 3^n$.

Section 8.4: Generating functions.

Taylor series to know and love:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + x^n \qquad \text{(finite)}$$

$$\frac{1-x^n}{1-x} = \sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} \qquad \text{(finite)}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \qquad \text{(infinite)}$$

$$e^x = \sum_{k=0}^{\infty} x^k/k! = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \qquad \text{(infinite)}$$

Combining series: Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x)+g(x) = \sum_{k=0}^{\infty} (a_k+b_k)x^k \quad \text{and} \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k.$$

Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.
Approach 1: Use the multiplication rule,

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i b_{k-i}\right) x^k,$$

on

$$\frac{1}{(1-x)^2} = \frac{1}{1-x} * \frac{1}{1-x} = \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{k=0}^{\infty} x^k\right).$$

Here, $a_i = b_i = 1$ for all i.
So

$$\sum_{i=0}^{k} a_i b_{k-i} = \sum_{i=0}^{k} 1 * 1 = k+1.$$

Thus

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.
Approach 2: Use derivatives, noting that

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}.$$

Thus,

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}(1-x)^{-1} = \frac{d}{dx}\sum_{k=0}^{\infty} x^k \\ &= \sum_{k=0}^{\infty}\frac{d}{dx}x^k \\ &= \sum_{k=0}^{\infty} kx^{k-1} \qquad \text{(change summation: let } j = k-1\text{)} \\ &= \sum_{j=0}^{\infty}(j+1)x^j = 1 + 2x + 3x^2 + 4x^3 + \cdots.\end{aligned}$$

You try: Exercise 36