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A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

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Solutions to this kind of relation come in the form r^n, where r is a root of the characteristic equation, which is obtained by plugging in r^n, dividing through by r^{n-k}, and solving for 0:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0.$$

The roots of this equation are called the characteristic roots.

Example from last time: plugging $a_n = r^n$ into the recursion relation

$$a_n = -a_{n-1} + a_{n-2} + a_{n-3}$$

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The characteristic roots are $r_1 = 1$ (with multiplicity 1) and $r_2 = -1$ (with multiplicity 2).

Theorem: Solving linear homogeneous recurrences

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation has roots $r_1, r_2, \ldots, r_\ell$ with multiplicities $m_1, m_2, \ldots, m_\ell$. Then a sequence $\{a_n\}_{n\in\mathbb{N}}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$a_n = p_1(n) r_1^n + p_2(n) r_2^n + \cdots + p_\ell(n) r_\ell^n,$$

where $p_i(n)$ are polynomials in n of degree $m_i - 1$.

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General solution:

$$a_n = (\alpha_0 + \alpha_1 n + \alpha_2 n^2)(-1)^n + (\beta_0 + \beta_1 n)(2)^n.$$

Non-homogeneous equations

Suppose your recurrence is linear and constant coefficient in a_i's, but is not homogeneous. In other words, it is in the form

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The following theorem says that if we can find *one solution* to a_n, then the general solutions to h_n will help us build all the rest of the solutions to a_n.

Theorem: Solving non-homogeneous equations

(a) If $\{\hat{a}_n\}_{n\in\mathbb{N}}$ is one solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n = \hat{a}_n + h_n\}_{n\in\mathbb{N}}$, where $\{h_n\}_{n\in\mathbb{N}}$ is a solution of the associated homogeneous recurrence relation.

(b) Finding $\hat{a}_n$: If $F(n) = Q(n)R^n$, where

- $Q(n)$ is a polynomial in n, and
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then there is a solution to a_n of the form

$$\hat{a}_n = n^m q(n) R^n$$

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- $\deg(q(n)) \leqslant \deg(Q(n))$, and
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2. Solve for h_n:
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3. Fine one solution $\hat{a}_n$ by guessing something of a similar form.

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General sol: $a_n = \alpha(-2)^n + (\alpha + \beta n)3^n + \frac{32}{3}4^n$.

1. Break the sequence into two parts: homogeneous h_n and a function of n: $\boxed{a_n = h_n + F(n)}$.
2. Solve for h_n:
   - compute the characteristic equation;
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3. Fine one solution $\hat{a}_n$ by guessing something of a similar form.

   If $F(n) = Q(n)R^n$, guess $\hat{a}_n = n^m q(n) R^n$

   where $m =$ mult of R, and $q_n = b_0 + b_1 n + b_2 n^2 + \cdots + b_d n^d$ where $d = \deg(Q(n))$.

Example: $a_n = 4a_{n-1} + 3a_{n-2} - 18a_{n-3} + n4^n$

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Char eq: $0 = r^3 - 4r^2 - 3r + 18$

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Char eq: $0 = r^3 - 4r^2 - 3r + 18 = (r+2)(r-3)^2$

Homog sol: $h_n = \alpha(-2)^n + (\alpha + \beta n)3^n$

Particular solution guess: $\hat{a}_n = (b_0 + b_1 n)4^n$

1. Break the sequence into two parts: homogeneous h_n and a function of n: $\boxed{a_n = h_n + F(n)}$.
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Particular solution guess: $\hat{a}_n = (b_0 + b_1 n)4^n$

(gives $b_0 = 1376/9$ and $b_1 = -63/3$)

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Particular solution guess: $\hat{a}_n = (b_0 + b_1 n)4^n$

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General sol: $a_n = \alpha(-2)^n + (\alpha + \beta n)3^n + (\frac{1376}{9} - \frac{63}{3}n)4^n$.

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Particular solution guess: $\hat{a}_n = bn^2 3^n$

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Homog sol: $h_n = \alpha(-2)^n + (\alpha + \beta n)3^n$

Particular solution guess: $\hat{a}_n = (b_0 + b_1 n)n^2 3^n$

(gives $b_0 = 21/50$ and $b_1 = 1/10$)

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General sol: $a_n = \alpha(-2)^n + (\alpha + \beta n)3^n + (\frac{21}{50} + \frac{1}{10}n)n^2 3^n$.

Section 8.4: Generating functions.

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Taylor series to know and love:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + x^n \qquad \text{(finite)}$$

$$\frac{1-x^n}{1-x} = \sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} \qquad \text{(finite)}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \qquad \text{(infinite)}$$

$$e^x = \sum_{k=0}^{\infty} x^k/k! = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \qquad \text{(infinite)}$$

Section 8.4: Generating functions.

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Combining series: Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x)+g(x) = \sum_{k=0}^{\infty} (a_k+b_k) x^k \quad \text{and} \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k.$$

Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.

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$$\frac{1}{(1-x)^2} = \frac{1}{1-x} * \frac{1}{1-x} = \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{k=0}^{\infty} x^k\right).$$

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You try: Exercise 36