Last time:

Using recurrence relations to model counting problems.

Today:

Solving those recurrence relations!

Try Warmup

8.2: Solving linear recurrence relations

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

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Examples:

1. $a_n = a_{n-1} + a_{n-2}$ (i.e. $c_1 = c_2 = 1$.)
2. $a_n = 3a_{n-1} - a_{n-3}$ (i.e. $c_1 = 3$, $c_2 = 0$, $c_3 = -1$.)

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Non-examples:

1. $a_n = a_{n-1}a_{n-2}$
2. $a_n = a_{n-1} + 1$
3. $a_n = a_{n-1}^2$
4. $a_n = na_{n-1}$

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Solving recurrences in general is hard (i.e. no deterministic way to do it). We take the same approach as in solving integrals and differential equations: look at the form the recurrence takes, make an educated guess, and solve for unknowns.

Analogy to differential equations

For those who have taken 391 - otherwise ignore this slide!

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$$y^{(k)} = c_1 y^{(k-1)} + c_2 y^{(k-2)} + \cdots + c_{k-2} y' + c_{k-1} y.$$

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Linearity further gives us that $y = a_1 y_1 + a_2 y_2$ is also a solution for any constants a_1 and a_2.

Back to solving recurrence relations: Educated guessing

Note that $a_n = r^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

if and only if (plug in $a_n = r^n$)

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So long as $r \neq 0$ (which is always a solution), this is equivalent to

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0 \qquad (*)$$

(subtract the RHS from both sides, and remove as many factors of r as possible).

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We call this last equation the characteristic equation for the recurrence relation (same as for differential equations). The solutions to this equation are the characteristic roots.

Char. eqn.: $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$

For example, consider the Fibonacci sequence:

$$\boxed{a_n = a_{n-1} + a_{n-2}.}$$

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Recall the quadratic formula:

$$ax^2 + bx + c = 0 \quad \text{if and only if } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

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$$r_1 = \tfrac{1}{2}\left(1 + \sqrt{5}\right) \quad \text{and} \quad r_2 = \tfrac{1}{2}\left(1 - \sqrt{5}\right)$$

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You try: plug in $a_n = r^n$ and simplify.

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$$r^3 + r^2 - r - 1 = r^2(r+1) - (r+1)$$

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So the characteristic roots are $r_1 = 1$ (with multiplicity 1) and $r_2 = -1$ (with multiplicity 2).

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You try: Exercise 31.

Theorem 1: Solving linear homogeneous with distinct roots

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}_{n \in \mathbb{N}}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

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for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

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Example: For $a_n = a_{n-1} + a_{n-2}$, we found characteristic roots

$$r_1 = \frac{1}{2}\left(1 + \sqrt{5}\right) \quad \text{and} \quad r_2 = \frac{1}{2}\left(1 - \sqrt{5}\right),$$

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each with multiplicity 1. So the general solution is

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Example: For $a_n = -a_{n-1} + a_{n-2} + a_{n-3}$, $r_2 = -1$ had multiplicity 2, so this theorem does not apply!

Incorporating initial conditions: specific solutions

Now, we have a general solution to the recurrence relation $a_n = a_{n-1} + a_{n-2}$:

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You try Exercise 32

Theorem 2: Solving linear homogeneous with repeated roots

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation has roots $r_1, r_2, \ldots, r_\ell$ with multiplicities $m_1, m_2, \ldots, m_\ell$. Then a sequence $\{a_n\}_{n\in\mathbb{N}}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

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Example: $a_n = 5a_{n-1} + 5a_{n-2} - 25a_{n-3} - 40a_{n-4} - 16a_{n-5}$.

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You try Exercise 33

Let's see why:

Proof of Theorem 1 for $k = 2$. Consider the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

and suppose the corresponding characteristic equation $r^2 - c_1 r - c_2 = 0$ has distinct roots r_1 and r_2.

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&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
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Substitute this into the second equation to get

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This completes our proof of Theorem 1 for $k = 2$.