Math 365 – Monday 3/11/19

(We're replacing the old #26.)

Exercise 26.

(a) Consider strings of length 10 consisting of 1's, 2's, and/or 3's.
  (i) How many of these are there (with no additional restrictions)?
  (ii) How many of these are there that contain exactly three 1's, two 2's, and five 3's?

(b) How many anagrams are there of MISSISSIPPI?

(c) Suppose you've got eight varieties of doughnuts to choose from at a doughnuts shop.
  (i) How many ways can you select 6 doughnuts?
  (ii) How many ways can you select a dozen (12) doughnuts?
  (iii) How many ways can you select a dozen doughnuts with at least one of each kind?
  [Hint: if there's at least one of each kind, then how many choices are you really making?]

(d) How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a jar contain if it has 20 coins in it?

(e) Counting solutions.
  (i) How many solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where x_1, x_2, and x_3 are nonnegative integers?
  (ii) How many solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where x_1, x_2, and x_3 are strictly positive integers?
  [Hint: See problem (c)(iii)]
  (iii) How many solutions are there to the equation $x_1 + x_2 + x_3 \leq 10$, where x_1, x_2, and x_3 are nonnegative integers? [Hint: Use an extra variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 10$]

Exercise 27.

(a) List the partitions of 6, both as box diagrams and as sequences.

(b) How many ways are there to distribute 6 identical cookies into 6 identical lunch boxes, possibly leaving some empty?

(c) How many ways are there to distribute 6 identical snack bars into 4 identical lunch boxes, possibly leaving some empty?

(d) How many ways are there to distribute 4 identical apples into 6 identical lunch boxes, possibly leaving some empty?

Exercise 28.

(a) Basic counting:
  (i) How many ways are there to distribute 5 distinguishable objects into 3 distinguishable boxes, possibly leaving some empty?
  (ii) How many ways are there to distribute 5 indistinguishable objects into 3 distinguishable boxes, possibly leaving some empty?
  (iii) How many ways are there to distribute 5 distinguishable objects into 3 indistinguishable boxes, possibly leaving some empty?
  (iv) How many ways are there to distribute 5 indistinguishable objects into 3 indistinguishable boxes, possibly leaving some empty?
  (v) How many ways are there to distribute 6 distinguishable objects into 4 indistinguishable boxes, possibly leaving some empty?
  (vi) How many ways are there to distribute 6 distinguishable objects into 4 indistinguishable boxes so that each of the boxes contains at least one object?

(b) How many ways are there to pack 8 identical DVDs into 5 indistinguishable boxes? How many ways to do this task so that each box contains at least one DVD?

(c) How many ways are there to distribute 5 balls into 7 boxes if

  (i) both the balls and boxes are labeled?

  (ii) the balls are labeled, but the boxes are unlabeled?

  (iii) the balls are unlabeled, but the boxes are labeled?

  (iv) both the balls and boxes are unlabeled?

(d) Repeat parts (i)–(iv) of part (c), adding the condition that each bucket can have at *most* one ball in it.

Warmup

1. How many 5-letter words are there?
2. How many ways possible outcome are there of ten flips of a coin?
3. How many ways ways can you answer a 20-question T or F quiz?

For the following, you'll either want to use division rule or $\binom{n}{k}$.

4. How many 5-character passwords are there with four 'A's, two 'B', and one 'C'?

Caution: The book talks about *permutation* problems and *combination* problems **with or without repetition or replacement**. When the rest of the world says permutation or combination problem without clarification, they only mean without repetition/replacement!

"Permutations with repetition":

i.e. "ordered selection with replacement"

Permutation means that order matters.
Repetition means you can repeat objects.

Example questions: 1-3 on the warmup.
Q. How many possible outcomes are there for drawing one card out of a deck at a time, recording its value and suite, and then replacing it, doing so 10 times?

Theorem. The number of ways to pick n objects, in order, with possible repetition, from a set of k objects is $\boxed{k^n}$.

$$\underbrace{\underline{(k)}\quad \underline{(k)}\quad \underline{(k)}\quad \underline{(k)}\quad \cdots\quad \underline{(k)}}_{n \text{ times}}$$

Permutations with indistinguishable objects.

Permutation means that order matters.
Indistinguishable means there are objects that can't be told apart.

Example questions: #4 on the warmup.

- How many anagrams are there of SUCCESS?
  Objects: 3 S's, 1 U, 2 C's, 1 E. Places: 7
  Place 3 S's: $\binom{7}{3}$, Place 1 U: $\binom{7-3}{1}$,
  Place 2 C's: $\binom{7-3-1}{2}$, Place 1 E: $\binom{7-3-1-2}{1}$.
  Total: $\boxed{\binom{7}{3}\binom{4}{1}\binom{3}{2}\binom{1}{1}}$

Solution strategy 1: Make a list of the objects and how many times they're used. Then place the objects one "type" at a time.

Permutations with indistinguishable objects.

In general: the number of ways to place n objects consisting of exactly

$$n_1 \text{ `}O_1\text{'s}, \quad n_2 \text{ `}O_2\text{'s}, \quad \ldots, \quad \text{and } n_k \text{ `}O_k\text{'s}$$

(so that $n = n_1 + \cdots + n_k$), in order is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_k}{n_k} \qquad \text{(since } n-(n_1+\cdots n_{k-1}) = n_k\text{)}.$$

Simplifying:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_k}{n_k}$$
$$= \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{n_k!}{n_k!0!}$$
$$= \boxed{\frac{n!}{n_1!n_2!n_3!\cdots n_k!}}. \qquad \text{(Solution 2)}$$

6.5: Generalized permutations and combinations continued

So far: Placing objects in boxes (combination problems)

"How many ways can you place n objects into k boxes, if...?"

For each question, ask yourself:

Can you tell the objects apart? Can you tell the boxes apart?

If you **can** tell them apart, we call them distinguishable.

(Objects: Cards face up. Boxes: labeled.)

If you **cannot** tell them apart, we call them indistinguishable.

(Objects: Cards face down. Boxes: unlabeled.)

We did: Distinguishable objects into distinguishable boxes.

More example questions:

(1) How many ways can you **evenly** distribute twelve articles to three editors to be reviewed?

(2) How many ways can you distribute twelve articles to three editors to be reviewed (if you don't care how many articles each person gets)?

(3) How many ways are there to distribute hands of 5 cards to each of 4 players from the standard deck of 52 cards?

(II) Indistinguishable objects into distinguishable boxes

(Place cards face down)

Example questions:

(1) How many outcomes can there be for the final tally in an election with 5 candidates and 100 voters?

(2) How many ways are there to pick a collection of four pieces of fruit from a bowl containing lots of apples, oranges, and pears?

(3) How many nonnegative integer solutions are there to the equation $x_1 + x_2 + x_3 = 11$?

For (1), the 100 votes are the objects and the 5 candidates are the boxes.

For (2), the 4 choices are the objects and the 3 fruit types are the boxes (think 'voting for fruit').

For (3), there are 11 objects to be placed into the 3 boxes labeled x_1, x_2, and x_3 (think 'voting for variables').

Strategy: "stars and bars"

(II) Indistinguishable objects into distinguishable boxes

Stars and bars. Like before, lay out the objects in a line. But now, we can't tell the difference between the objects. So instead of naming them, just represent them using stars:

$$\underbrace{* * * * * \cdots * * *}_{n \text{ objects}}$$

Next, partition them into the k boxes using bars:

$$\underbrace{\overbrace{* * \cdots *}^{\substack{\text{objects in} \\ \text{1st box}}} \;\Big|\; \overbrace{* * \cdots *}^{\substack{\text{objects in} \\ \text{2nd box}}} \;\Big|\; \cdots \;\Big|\; \overbrace{* * \cdots *}^{\substack{\text{objects in} \\ k\text{th box}}}}_{\substack{n \text{ stars and } k-1 \text{ bars gives} \\ n+(k-1) \text{ symbols in total}}}$$

three in box 1
one in box 2
two in box 3

two in box 1
four in box 2
none in box 3

six in box 1
none in box 2
none in box 3

(II) Indistinguishable objects into distinguishable boxes

Stars and bars. Like before, lay out the objects in a line. But now, we can't tell the difference between the objects. So instead of naming them, just represent them using stars:

$$\underbrace{* * * * * \cdots * **}_{n \text{ objects}}$$

Next, partition them into the k boxes using bars:

$$\underbrace{\overbrace{* * \cdots *}^{\substack{\text{objects in}\\ \text{1st box}}} \mid \overbrace{* * \cdots *}^{\substack{\text{objects in}\\ \text{2nd box}}} \mid \quad \cdots \quad \mid \overbrace{* * \cdots *}^{\substack{\text{objects in}\\ k\text{th box}}}}_{\substack{n \text{ stars and } k-1 \text{ bars gives}\\ n+(k-1) \text{ symbols in total}}}$$

So we're down to counting **anagrams** of n stars and $k-1$ bars:

$$\boxed{\frac{(n+(k-1))!}{n!(k-1)!}}$$

(II) Indistinguishable objects into distinguishable boxes

Theorem. The number of ways to distribute n indistinguishable objects amongst k distinguishable boxes is

$$\boxed{\frac{(n+(k-1))!}{n!(k-1)!}}$$

Back to example questions:

(1) How many outcomes can there be for the final tally in an election with 5 candidates and 100 voters?

Answer: the 100 votes are the objects $(n = 100)$ and the 5 $(k = 5)$ candidates are the boxes. So the number of possible outcomes is

$$\boxed{\frac{(100+(5-1))!}{100!(5-1)!} = \frac{104!}{100!4!}}$$

(II) Indistinguishable objects into distinguishable boxes

Theorem. The number of ways to distribute n indistinguishable objects amongst k distinguishable boxes is

$$\boxed{\frac{(n+(k-1))!}{n!(k-1)!}}$$

Back to example questions:
(2) How many ways are there to pick a collection of four pieces of fruit from a bowl containing lots of apples, oranges, and pears?

Answer: the 4 choices are the objects ($n = 4$) and the 3 fruit types are the boxes (think 'voting for fruit') ($k = 3$) . So the number of possible outcomes is

$$\boxed{\frac{(4+(3-1))!}{4!(3-1)!} = \frac{6!}{4!2!}}$$

(II) Indistinguishable objects into distinguishable boxes

Theorem. The number of ways to distribute n indistinguishable objects amongst k distinguishable boxes is

$$\boxed{\frac{(n+(k-1))!}{n!(k-1)!}}$$

Back to example questions:
(3) How many nonnegative integer solutions are there to the equation $x_1 + x_2 + x_3 = 11$?

Answer: there are 11 objects ($n = 11$) to be placed into the 3 boxes ($k = 3$) labeled x_1, x_2, and x_3 (think 'voting for variables'). So the number of possible outcomes is

$$\boxed{\frac{(11+(3-1))!}{11!(3-1)!} = \frac{13!}{11!2!}}$$

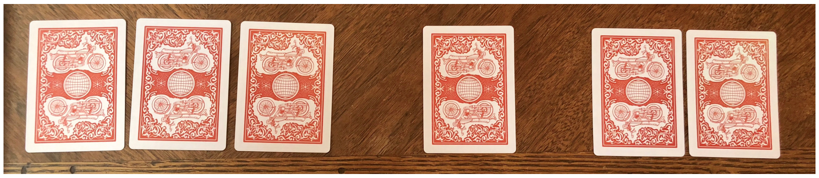
(See Exercise 26.)

(III) Indistinguishable objects into indistinguishable boxes

Example question: How many ways can you distribute 4 apples into three unmarked baskets?
(Specifically, the possibilities you're counting are like "all four apples go into one basket", or "one basket has 2 apples, and the other two each has 1 apple", and so on.)

Instead of. . .

we're counting things like. . .



(III) Indistinguishable objects into indistinguishable boxes

Example question: How many ways can you distribute 4 apples into three unmarked baskets?
(Specifically, the possibilities you're counting are like "all four apples go into one basket", or "one basket has 2 apples, and the other two each has 1 apple", and so on.)

Also note that (in the indistinguishable boxes case) . . .

is the same as. . .

(III) Indistinguishable objects into indistinguishable boxes

Example question: How many ways can you distribute 4 apples into three unmarked baskets?

Hard: we can't line anything up anymore, since there's no 1st, 2nd, etc.. By "hard", I mean there's no closed formula.

Rearrange all possible outcomes from most full basket to least full:
4 in one basket;
3 in one, 1 in another;
2 in one, 2 in another;
2 in one, 1 in another, 1 in another;
~~1 in each~~.

An (integer) partition of a positive integer n is a way of breaking n into whole pieces, without order. Alternatively, a partition λ of n, written $\lambda \vdash n$ is a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

satisfying

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell, \quad \lambda_i \in \mathbb{Z}_{>0} \text{ and } \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell.$$

An (integer) partition of a positive integer n is a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

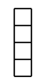
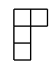
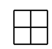
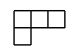
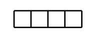
satisfying

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell, \quad \lambda_i \in \mathbb{Z}_{>0} \text{ and } \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell.$$

Partitions are hard to count (i.e. there is **no closed formula**) – you have to do it manually. Easier to count if we draw!

Draw partitions as n boxes piled up and to the left into a corner, where the ith row has λ_i boxes.

For example,

the partition	$(1,1,1,1)$	is	 ;
the partition	$(2,1,1)$	is	 ;
the partition	$(2,2)$	is	 ;
the partition	$(3,1)$	is	 ;
and the partition	(4)	is	 .

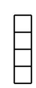
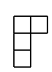
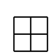
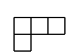
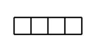
An (integer) partition of a positive integer n is a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

satisfying

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell, \quad \lambda_i \in \mathbb{Z}_{>0} \text{ and } \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell.$$

For example,

the partition	$(1,1,1,1)$	is	 ;
the partition	$(2,1,1)$	is	 ;
the partition	$(2,2)$	is	 ;
the partition	$(3,1)$	is	 ;
and the partition	(4)	is	 .

The entries in the sequence are called the parts; λ_i is the length of the ith part. Denote

the number of partitions of n with are most k parts	by $p_k(n)$;
and the number of partitions of n	by $p(n)$.

(IV) Distinguishable objects into indistinguishable boxes

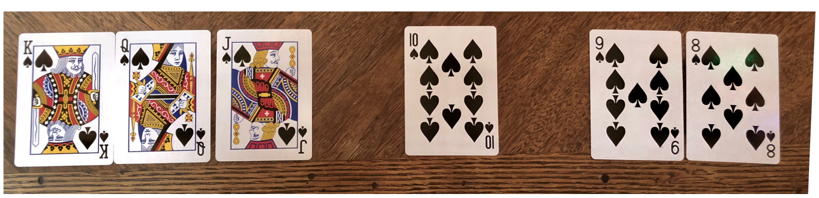
Example question: How many ways are there to put n different employees into k basically identical offices, when each office can contain any number of employees?

Instead of...

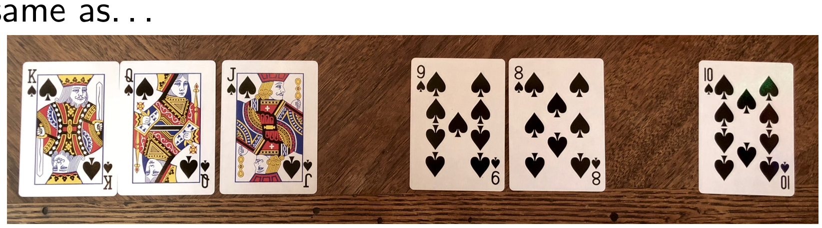
we're counting things like...

(IV) Distinguishable objects into indistinguishable boxes

Also note that (in the indistinguishable boxes case) ...



is the same as...



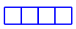
but not...

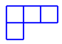
(since the objects are distinguishable).

(IV) Distinguishable objects into indistinguishable boxes

Example question: How many ways are there to put four different employees into three basically identical offices, when each office can contain any number of employees?

For small examples: Think of how to partition $\{A, B, C, D\}$ into up to three subsets:

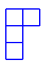
 (all in one set): $\{A, B, C, D\}$

 (three in one, one alone): 4 of these–choose who's alone

$\{A, B, C\} \sqcup \{D\}$, $\{A, B, D\} \sqcup \{C\}$,
$\{A, C, D\} \sqcup \{B\}$, or $\{B, C, D\} \sqcup \{A\}$.

 (two and two): 3 of these–choose A's officemate

$\{A, B\} \sqcup \{C, D\}$, $\{A, C\} \sqcup \{B, D\}$, or $\{A, D\} \sqcup \{B, C\}$

 (two, one, and one): $\binom{4}{2}$ of these–choose who's alone

$\{A, B\} \sqcup \{C\} \sqcup \{D\}$, $\{A, C\} \sqcup \{B\} \sqcup \{D\}$, $\{A, D\} \sqcup \{B\} \sqcup \{C\}$,
$\{B, C\} \sqcup \{A\} \sqcup \{D\}$, $\{B, D\} \sqcup \{A\} \sqcup \{C\}$, $\{C, D\} \sqcup \{A\} \sqcup \{D\}$

Total: $1 + 4 + 3 + 6 = 14$.

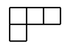
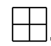
(IV) Distinguishable objects into indistinguishable boxes

Stirling numbers of the second kind: Let $S(n,j)$ be the number of ways to distribute n distinguishable things into **exactly** j boxes (so that none of the boxes are empty). One can use inclusion/exclusion to find (We will not prove this yet)

$$S(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n.$$

Example: In our example of assigning people to offices...
$S(4,1)$ counts putting everyone into one office:

$$S(4,1) = \frac{1}{1!}\sum_{i=0}^{1-1}(-1)^i\binom{1}{i}(1-i)^4 = (-1)^0\binom{1}{0}(1-0)^4 = 1.\checkmark$$

$S(4,2)$ counts putting people into exactly 2 offices, i.e. the cases corresponding to the partitions with 2 parts,  and .

$$S(4,2) = \frac{1}{2!}\sum_{i=0}^{2-1}(-1)^i\binom{2}{i}(2-i)^4 = \frac{1}{2}\left(\binom{2}{0}2^4 - \binom{2}{1}1^4\right) = 7 = 3+4.\checkmark$$

(IV) Distinguishable objects into indistinguishable boxes

Stirling numbers of the second kind: Let $S(n,j)$ be the number of ways to distribute n distinguishable things into exactly j boxes (so that none of the boxes are empty). One can use inclusion/exclusion to find (We will not prove this)

$$S(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n.$$

Then the number of ways to distribute n distinguishable things into k boxes (when we don't care if some are left empty) is

$$\sum_{j=1}^{k} S(n,j) = S(n,1) + S(n,2) + \cdots + S(n,k).$$

Example: In our example of assigning people to offices, we should get

$$\sum_{j=1}^{3} S(4,j) = S(4,1) + S(4,2) + S(4,3) = 1 + 7 + 6 = 14.\checkmark$$

Summary of counting techniques:

Placing objects in order ("permutation")

With replacement: The number of ways to pick n objects, in order, with possible repetition, from a set of k objects is $\boxed{k^n}$.

With some indistinguishable objects (anagrams): The number of ways to place n objects consisting of exactly

$$n_1 \text{ ‘}O_1\text{'s}, \quad n_2 \text{ ‘}O_2\text{'s}, \quad \ldots, \quad \text{and } n_k \text{ ‘}O_k\text{'s}$$

(so that $n = n_1 + \cdots + n_k$), in order is

$$\boxed{\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_r}{n_r} = \frac{n!}{n_1!n_2!n_3!\cdots n_k!}.}$$

Placing objects into boxes ("combination": no order inside the boxes)

Distinguishable objects, distinguishable boxes: Distributing n distinguishable objects into k distinguishable boxes is the same as the permutation problems above. If there are no restrictions, then $\boxed{k^n}$. If you restrict to placing exactly n_i objects into box i, for $i = 1, 2, \ldots, k$ (so that $n = n_1 + \cdots + n_r$), is

$$\boxed{\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_k}{n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}.}$$

Indistinguishable objects, distinguishable boxes: (*Stars and bars*) The number of ways to distribute n indistinguishable objects into k distinguishable boxes is

$$\boxed{\binom{n+k-1}{n} = \binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!(k-1)!}.}$$

Indistinguishable objects, indistinguishable boxes: (*Integer partitions*) The number of ways to distribute n indistinguishable objects into k indistinguishable boxes is the same of the number of integer partitions of n into at most k parts, $\boxed{p_k(n)}$ (there is no closed formula; you just have to count them).

Distinguishable objects, indistinguishable boxes: The number of ways to distribute n distinguishable objects into k indistinguishable boxes is given by

$$\boxed{\sum_{j=1}^{k} S(n,j), \qquad \text{where} \qquad S(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n.}$$

We call the numbers given by $S(n,j)$ the *Serling numbers of the second kind.*

A note on conditions like "where there's at least one of each kind", or "where there's at least one object in each box", or solving linear equations using *strictly positive* values: these conditions effectively just decrease the number of choices you're making.

Example: Choose 10 pieces of fruit from a bowl with indistinguishable apples, oranges, and bananas, making sure to choose at least one of each kind.

Answer: Effectively, you've already picked out three pieces of fruit: one apple, one orange, and one pear. So you only need to count how many ways you can make the remaining $10-3$ choices, for which you will use stars and bars, with $n = 10-3 = 7$ and $k = 3$ (the number of kinds of fruit).