Warmup

1. How many 5-letter words are there?
2. How many ways possible outcome are there of ten flips of a coin?
3. How many ways ways can you answer a 20-question T or F quiz?

For the following, you'll either want to use division rule or $\binom{n}{k}$.

4. How many 5-character passwords are there with four 'A's, two 'B', and one 'C'?

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1. How many 5-letter words are there? $\boxed{26^5}$
2. How many ways possible outcome are there of ten flips of a coin? $\boxed{2^{10}}$
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   Solution 2. First place the all 7 letters in order: $7!$.
   Then divide by the rearrangements of the four A's: $4!$
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i.e. "ordered selection with replacement"

Permutation means that order matters.

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Example questions: 1-3 on the warmup.
Q. How many possible outcomes are there for drawing one card out of a deck at a time, recording its value and suite, and then replacing it, doing so 10 times?

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Example questions: 1-3 on the warmup.
Q. How many possible outcomes are there for drawing one card out of a deck at a time, recording its value and suite, and then replacing it, doing so 10 times?

Theorem. The number of ways to pick n objects, in order, with possible repetition, from a set of k objects is $\boxed{k^n}$.

$$\underbrace{\underline{\ (k)\ }\quad \underline{\ (k)\ }\quad \underline{\ (k)\ }\quad \underline{\ (k)\ }\quad \dots\quad \underline{\ (k)\ }}_{n \text{ times}}$$

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Solution strategy 1: Make a list of the objects and how many times they're used. Then place the objects one "type" at a time.

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In general: the number of ways to place n objects consisting of exactly

$$n_1 \text{ `}O_1\text{'s}, \quad n_2 \text{ `}O_2\text{'s}, \quad \ldots, \quad \text{and } n_k \text{ `}O_k\text{'s}$$

(so that $n = n_1 + \cdots + n_k$), in order is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_k}{n_k} \qquad \text{(since } n-(n_1+\cdots n_{k-1}) = n_k\text{).}$$

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Simplifying:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_k}{n_k}$$
$$= \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{n_k!}{n_k!0!}$$

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$$= \boxed{\frac{n!}{n_1!n_2!n_3!\cdots n_k!}}. \qquad \text{(Solution 2)}$$

6.5: Generalized permutations and combinations continued

So far: Placing objects in boxes (combination problems)

"How many ways can you place n objects into k boxes, if. . . ?"

For each question, ask yourself:

Can you tell the objects apart? Can you tell the boxes apart?

If you **can** tell them apart, we call them distinguishable.

(Objects: Cards face up. Boxes: labeled.)

If you **cannot** tell them apart, we call them indistinguishable.

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We did: Distinguishable objects into distinguishable boxes.

More example questions:

(1) How many ways can you **evenly** distribute twelve articles to three editors to be reviewed?

(2) How many ways can you distribute twelve articles to three editors to be reviewed (if you don't care how many articles each person gets)?

(3) How many ways are there to distribute hands of 5 cards to each of 4 players from the standard deck of 52 cards?

(II) Indistinguishable objects into distinguishable boxes

(Place cards face down)

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Example questions:

(1) How many outcomes can there be for the final tally in an election with 5 candidates and 100 voters?

(2) How many ways are there to pick a collection of four pieces of fruit from a bowl containing lots of apples, oranges, and pears?

(3) How many nonnegative integer solutions are there to the equation $x_1 + x_2 + x_3 = 11$?

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(3) How many nonnegative integer solutions are there to the equation $x_1 + x_2 + x_3 = 11$?

For (1), the 100 votes are the objects and the 5 candidates are the boxes.
For (2), the 4 choices are the objects and the 3 fruit types are the boxes (think 'voting for fruit').
For (3), there are 11 objects to be placed into the 3 boxes labeled x_1, x_2, and x_3 (think 'voting for variables').

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Strategy: "stars and bars"

(II) Indistinguishable objects into distinguishable boxes

Stars and bars. Like before, lay out the objects in a line. But now, we can't tell the difference between the objects. So instead of naming them, just represent them using stars:

$$\underbrace{* * * * * \cdots * **}_{n \text{ objects}}$$

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Next, partition them into the k boxes using bars:

$$\overbrace{* * \cdots *}^{\substack{\text{objects in} \\ \text{1st box}}} \mid \overbrace{* * \cdots *}^{\substack{\text{objects in} \\ \text{2nd box}}} \mid \quad \cdots \quad \mid \overbrace{* * \cdots *}^{\substack{\text{objects in} \\ k\text{th box}}}$$

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three in box 1
one in box 2
two in box 3

two in box 1
four in box 2
none in box 3

six in box 1
none in box 2
none in box 3

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So we're down to counting **anagrams** of n stars and $k-1$ bars:

$$\boxed{\frac{(n+(k-1))!}{n!(k-1)!}}$$

(II) Indistinguishable objects into distinguishable boxes

Theorem. The number of ways to distribute n indistinguishable objects amongst k distinguishable boxes is

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Back to example questions:

(1) How many outcomes can there be for the final tally in an election with 5 candidates and 100 voters?

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Back to example questions:

(1) How many outcomes can there be for the final tally in an election with 5 candidates and 100 voters?

Answer: the 100 votes are the objects $(n = 100)$ and the 5 $(k = 5)$ candidates are the boxes.

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Theorem. The number of ways to distribute n indistinguishable objects amongst k distinguishable boxes is

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Back to example questions:

(1) How many outcomes can there be for the final tally in an election with 5 candidates and 100 voters?

Answer: the 100 votes are the objects $(n = 100)$ and the 5 $(k = 5)$ candidates are the boxes. So the number of possible outcomes is

$$\boxed{\frac{(100+(5-1))!}{100!(5-1)!} = \frac{104!}{100!4!}}$$

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Theorem. The number of ways to distribute n indistinguishable objects amongst k distinguishable boxes is

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Back to example questions:

(2) How many ways are there to pick a collection of four pieces of fruit from a bowl containing lots of apples, oranges, and pears?

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Back to example questions:

(2) How many ways are there to pick a collection of four pieces of fruit from a bowl containing lots of apples, oranges, and pears?

Answer: the 4 choices are the objects $(n = 4)$ and the 3 fruit types are the boxes (think 'voting for fruit') $(k = 3)$.

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Back to example questions:

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Answer: the 4 choices are the objects $(n = 4)$ and the 3 fruit types are the boxes (think 'voting for fruit') $(k = 3)$. So the number of possible outcomes is

$$\boxed{\frac{(4+(3-1))!}{4!(3-1)!} = \frac{6!}{4!2!}}$$

(II) Indistinguishable objects into distinguishable boxes

Theorem. The number of ways to distribute n indistinguishable objects amongst k distinguishable boxes is

$$\boxed{\frac{(n+(k-1))!}{n!(k-1)!}}$$

Back to example questions:

(3) How many nonnegative integer solutions are there to the equation $x_1 + x_2 + x_3 = 11$?

(II) Indistinguishable objects into distinguishable boxes

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Back to example questions:

(3) How many nonnegative integer solutions are there to the equation $x_1 + x_2 + x_3 = 11$?

Answer: there are 11 objects ($n = 11$) to be placed into the 3 boxes ($k = 3$) labeled x_1, x_2, and x_3 (think 'voting for variables').

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Answer: there are 11 objects ($n = 11$) to be placed into the 3 boxes ($k = 3$) labeled x_1, x_2, and x_3 (think 'voting for variables'). So the number of possible outcomes is

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(See Exercise 26.)

(III) Indistinguishable objects into indistinguishable boxes

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Example question: How many ways can you distribute 4 apples into three unmarked baskets?

(Specifically, the possibilities you're counting are like "all four apples go into one basket", or "one basket has 2 apples, and the other two each has 1 apple", and so on.)

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Example question: How many ways can you distribute 4 apples into three unmarked baskets?
(Specifically, the possibilities you're counting are like "all four apples go into one basket", or "one basket has 2 apples, and the other two each has 1 apple", and so on.)

Instead of...

we're counting things like...

(III) Indistinguishable objects into indistinguishable boxes

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Also note that (in the indistinguishable boxes case) . . .

is the same as. . .

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Hard: we can't line anything up anymore, since there's no 1st, 2nd, etc..

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Hard: we can't line anything up anymore, since there's no 1st, 2nd, etc.. By "hard", I mean there's no closed formula.

(III) Indistinguishable objects into indistinguishable boxes

Example question: How many ways can you distribute 4 apples into three unmarked baskets?
(Specifically, the possibilities you're counting are like "all four apples go into one basket", or "one basket has 2 apples, and the other two each has 1 apple", and so on.)

Hard: we can't line anything up anymore, since there's no 1st, 2nd, etc.. By "hard", I mean there's no closed formula.

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(III) Indistinguishable objects into indistinguishable boxes

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An (integer) partition of a positive integer n is a way of breaking n into whole pieces, without order. Alternatively, a partition λ of n, written $\lambda \vdash n$ is a sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$$

satisfying

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell, \quad \lambda_i \in \mathbb{Z}_{>0} \text{ and } \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell.$$

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3 in one, 1 in another;	$\lambda = (3, 1)$
2 in one, 2 in another;	$\lambda = (2, 2)$
2 in one, 1 in another, 1 in another;	$\lambda = (2, 1, 1)$
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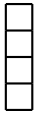
Partitions are hard to count (i.e. there is **no closed formula**) – you have to do it manually. Easier to count if we draw!
Draw partitions as n boxes piled up and to the left into a corner, where the ith row has λ_i boxes.
For example,

the partition $(1,1,1,1)$ is  ;

the partition $(2,1,1)$ is  ;

the partition $(2,2)$ is  ;

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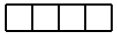
and the partition (4) is  .

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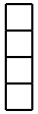
the partition	$(1,1,1,1)$	is	 ;
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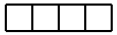
and the partition	(4)	is	 .

The entries in the sequence are called the parts; λ_i is the length of the ith part. Denote

the number of partitions of n with are most k parts	by $p_k(n)$;
and the number of partitions of n	by $p(n)$.

(IV) Distinguishable objects into indistinguishable boxes

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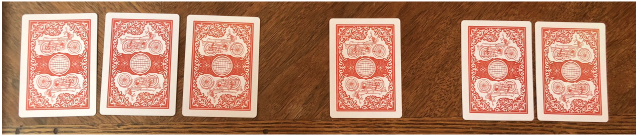
Instead of. . .

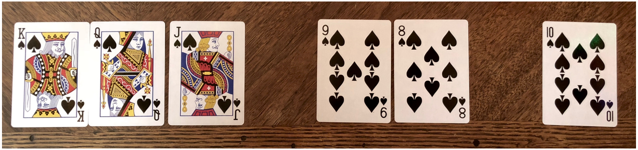


we're counting things like. . .

(IV) Distinguishable objects into indistinguishable boxes

Also note that (in the indistinguishable boxes case) . . .

is the same as. . .



but not. . .

(since the objects are distinguishable).

(IV) Distinguishable objects into indistinguishable boxes

Example question: How many ways are there to put four different employees into three basically identical offices, when each office can contain any number of employees?

For small examples: Think of how to partition $\{A, B, C, D\}$ into up to three subsets:

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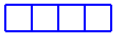 (all in one set): $\{A, B, C, D\}$

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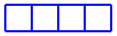 (all in one set): $\{A, B, C, D\}$

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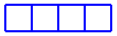 (all in one set): $\{A, B, C, D\}$

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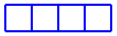 (all in one set): $\{A, B, C, D\}$

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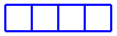 (all in one set): $\{A, B, C, D\}$

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Total: $1 + 4 + 3 + 6 = 14$.

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Then the number of ways to distribute n distinguishable things into k boxes (when we don't care if some are left empty) is

$$\sum_{j=1}^{k} S(n, j) = S(n, 1) + S(n, 2) + \cdots + S(n, k).$$

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Example: In our example of assigning people to offices, we should get

$$\sum_{j=1}^{3} S(4, j) = S(4, 1) + S(4, 2) + S(4, 3) = 1 + 7 + 6 = 14. \checkmark$$