**Math 365 – Wednesday 2/27/18 - 6.4: Binomial coefficients and identities**

Recall

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Tips for being slick:

(1) The values like $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{n} = 1$ are totally straightforward, and should not involve writing down formulas with factorials. In fact, you're more likely to make mistakes with the algebra than if you step back and ask yourself "How many ways can I pick one thing from 5?", "…nothing from 5?", "…everything from 5?"

(2) When doing computations, it's often easier to pre-cancel $(n-k)!$ or $k!$ from $n!$. For example,

$$\binom{11}{3} = \frac{(11!/(11-3)!)}{3!} = \frac{11*10*9}{3*2*1} = 11*5*3 = 165.$$

However, if $k$ is bigger than half of $n$, then use $k!$ to cancel terms from $n!$ instead. For example,

$$\binom{11}{8} = \frac{(11!/8!)}{(11-8)!} = \frac{11*10*9}{3*2*1} = 165,$$

rather than writing out, say,

$$\binom{11}{8} = \frac{(11!/(11-8)!)}{8!} = \frac{11*10*9*8*7*6*5*4}{8*7*6*5*4*3*2*1}.$$

**Warmup.** <u>Without a calculator</u>, compute the following values:

$\binom{4}{2} =$

$\binom{5}{3} =$

$\binom{8}{6} =$

$\binom{10}{3} =$

$\binom{10}{9} =$

$\binom{15}{0} =$

$\binom{200}{1} =$

$\binom{200}{200} =$

**Exercise 24.**

(a) Expand $(x+y)^5$ and $(x+y)^8$ using the binomial theorem.

(b) Substitute $x = 2z$ and $y = 3$ to calculate $(2z+3)^4$.

(c) What identity do you get if you substitute $x = -1$ and $y = 1$ in the binomial theorem? Check your identity (like in the previous problem) for $n = 4$.

(d) Using the binomial theorem to prove combinatorial identities.

  (i) Use the binomial theorem to explain why

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

  Then check and examples of this identity by calculating both sides for $n = 4$. (Hint: substitute $x = y = 1$).

  (ii) Use the binomial theorem to explain why

$$2^n = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-3)^k.$$

  Then check and examples of this identity by calculating both sides for $n = 4$. (Hint: what other examples can you think of of integers that sum to 2?).

(e) Give a formula for the coefficient of $x^k$ in the expansion of $(x + 1/x)^{100}$ , where $k$ is an integer.

(f) As a useful counting tool, we have (so far) only defined $\binom{n}{k}$ for non-negative integers $n$ and $k$. But in §8.4 (p. 539, Def. 2), the book defines "extended binomial coefficients" as follows: Let $u \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$. The *extended binomial coefficient* $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)(u-2)\cdots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{if } k = 0. \end{cases}$$

For example,

$$\binom{0.2}{3} = \frac{0.2(-0.8)(-1.8)}{3*2*1}.$$

  (i) Compute $\binom{\pi}{4}$, $\binom{1/2}{2}$, and $\binom{7/3}{0}$.

  (ii) Verify algebraically that if $n$ is a positive integer and $0 \leq k \leq n$, then

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

  Check this identity for $n = 5$ and $k = 3$ (you should probably do this first).

  (iii) BONUS: The *extended binomial theorem* states that for any real number $u$, we have

$$(1+x)^u = \sum_{i=0}^{\infty} \binom{u}{k} x^k.$$

  Now recall from calculus the Taylor series expansion

$$(1+x)^{-1} = \sum_{i=0}^{\infty} x^i (-1)^i.$$

  Check that the first 3 terms ($i = 0, 1, 2$) of our known Taylor series expansion match the first 3 terms of the extended binomial theorem expansion (when $u = -1$). Finally, verify that this example matches correctly for all terms by showing that $\binom{-1}{k} = (-1)^k$ for any $k \in \mathbb{Z}_{\geq 0}$.

  (iv) BONUS: Show in that the Taylor series expansion for $(1+x)^{-n}$ matches the extended binomial theorem for $n = 3$.

**Exercise 25.**

(a) Explain the example provided for the proof of Vandermonde's in the notes using words.

(b) Substitute $m = r = n$ into Vandermonde's identity to show that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2,$$

and check this identity for $n = 2$.

(c) Consider the identity

$$\binom{n}{k} k = \binom{n-1}{k-1} n$$

for integers $1 \le k \le n$.

(i) Verify this identity for $n = 5$ and $k = 3$.

(ii) Explain why this identity is true using a combinatorial argument.
[*Hint: Count, in two different ways, the number of ways to pick a subset with $k$ elements from a set with $n$ elements, along with a distinguished element of that $k$-element subset. For example, out of $n$ people, pick a committee of $k$ people and choose someone on that committee to organize their meetings.* ]

(iii) Illustrate your combinatorial proof using the set $A = \{a, b, c\}$ (so that $n = 3$) and $k = 2$.

(iv) Verify the identity algebraically using the formula $\binom{n}{k} = n!/((n-k)!k!)$.

**Exercise 26.**

(a) Consider strings of length 10 consisting of 1's, 2's, and/or 3's.

(i) How many of these are there?

(ii) How many of these are there that contain exactly three 1's, two 2's, and five 3's?

(iii) How many of these are there that contain at least three 1's, and exacly four 2's?

(b) How many anagrams are there of MISSISSIPPI?

Recall, on Day 1, we talked about expanding $(x+y)^n$:
The term $x^k y^{n-k}$ appears in

$$(x+y)^n = (x+y)(x+y)\cdots(x+y)$$

by picking $k$ terms to take $x$ from and then take $y$ from the remaining $n-k$ terms. For example, $x^2y$ appears $\binom{3}{2}$ times:

$$\begin{array}{cccccccc} & (x+y) & * & (x+y) & * & (x+y) & & \\ = & x & * & x & * & x & & \\ + & x & * & x & * & y & & \leftarrow \\ + & x & * & y & * & x & & \leftarrow \\ + & x & * & y & * & y & & \\ + & y & * & x & * & x & & \leftarrow \\ + & y & * & x & * & y & & \\ + & y & * & y & * & x & & \\ + & y & * & y & * & y & & \end{array}$$

So, the coefficient of $x^k y^{n-k}$ in $(x+y)^n$ is $\binom{n}{k} = \binom{n}{n-k}$.

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## Theorem

*Let $x$ and $y$ be variables, and let $n$ be a nonnegative integer. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This is our first binomial identity.

Example: In the expansion of $(x+y)^4$, the coefficient of

$\ldots x^4y^0 = x^4$ is $\binom{4}{0} = 1,\quad \ldots x^3y$ is $\binom{4}{1} = 4,$

$\ldots x^2y^2$ is $\binom{4}{2} = 6,\quad \ldots xy^3$ is $\binom{4}{3} = 4,$

$\ldots x^0y^4 = y^4$ is $\binom{4}{4} = 1,$

so

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Example: In the expansion of $(x+y)^{24}$, the coefficient of $x^{21}y^3$ is

$$\binom{21}{3} = \frac{21*20*19}{3*2*1}.$$

There are many ways to arrive at a "combinatorial identity".
One way is to count a set in two different ways.

Example: How many ways can you answer an $n$-question T or F quiz, if you might leave some answers blank?

Method 1: there are $n$ steps, at each step, there are 3 possible outcomes, so the answer is $3^n$.

Method 2: Choose the questions to leave blank (break into cases based on exactly how many to leave blank), and then answer the rest, so the answer is $\sum_{k=0}^{n} \binom{n}{k} 2^k$.

So

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Another way is to substitute special values into identities we already have.

Example: Substitute $x = 2, y = 1$ into

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}:$$

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Another way is to substitute special values into identities we already have.

Example: Substitute $x = 2, y = 1$ into

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}:$$

LHS: $(2+1)^n = 3^n$ RHS: $\sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^k.$

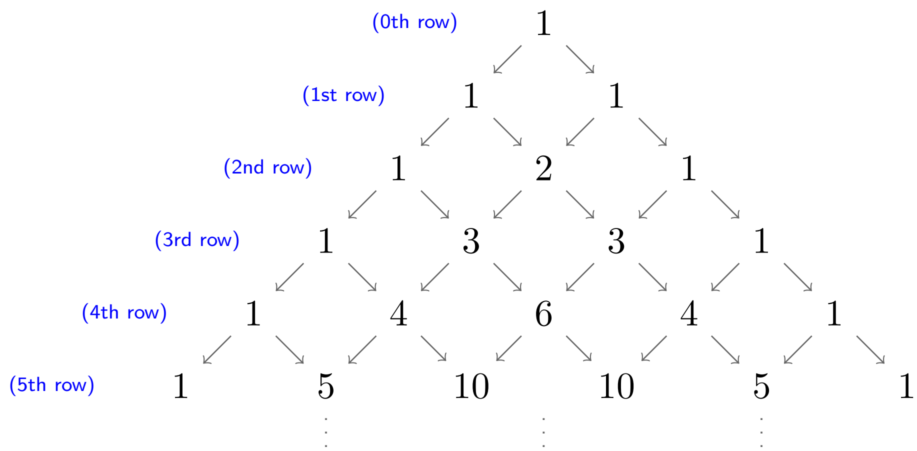
So

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

What if I had substituted $x = 1$ and $y = 2$ instead? I *should* get the same identity. (Use the fact that $\binom{n}{k} = \binom{n}{n-k}$.) See Ex 24.

## Recall Pascal's Triangle

Start and end each row with a 1. The $i$th row (starting with the 0th row) has $i+1$ entries. The middle entries are acquired by adding successive entries in the previous row.




Day 1 we argued (informally) $\binom{n}{k}$ is the $k$th entry of the $n$th row of Pascal's triangle

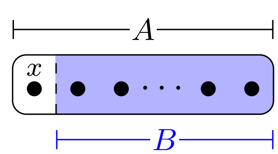
### Theorem (Pascal's identity)

*Let $n$ and $k$ be positive integers with $n \geqslant k$. Then*

> Theorem (Pascal's identity). Let $n$ and $k$ be positive integers with $n \geqslant k$. Then
> $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

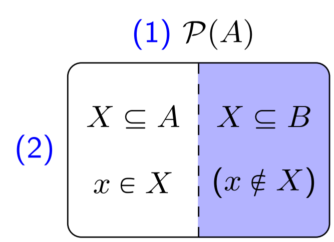
### **Combinatorial** proof of Pascal's identity:

We prove this identity "combinatorially", meaning we come up with a story about counting a set in two different ways. Then we'll conclude that since the two answers will count elements in the same set, they must be equal.




Let $A$ be a set of size $n$.
Fix $x \in A$ and let $B = A - \{x\}$.

Count subsets of $A$ of size $k$ in two ways:
(1) all at once, and
(2) broken into cases, depending on whether a subset contains $x$ or not.



**Theorem (Pascal's identity).** Let $n$ and $k$ be positive integers with $n \geqslant k$. Then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

**<u>Combinatorial</u> proof of Pascal's identity:**

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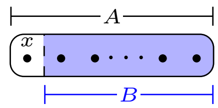



Count size-$k$ subsets of $A$ in 2 ways:

LHS, All at once: There are $\binom{n}{k}$ subsets of $A$ of size $k$.

RHS, Break into cases:

- <u>Case 1:</u> $x \notin X$. Since

$$\{X \subseteq A \mid x \notin X\} = \{X \subseteq B\} \quad \text{and} \quad |B| = n-1,$$

there are $\binom{n-1}{k}$ size-$k$ subsets of $A$ not containing $x$.

- <u>Case 2:</u> $x \in X$. Every size-$k$ subset of $A$ containing $x$ looks like $X = \{x\} \cup Y$, where $Y \subseteq B$ and $|Y| = k-1$. There are $\binom{n-1}{k-1}$ of these.

So, using the sum rule, there are $\binom{n-1}{k} + \binom{n-1}{k-1}$ subsets of $A$ of size $k$.

Conclusion: Thus $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. □

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Alternatively, we could prove this algebraically.

## <u>Algebraic</u> proof of Pascal's identity.

Recall,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

So, algebraically,

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

$$= \frac{(n-1)!}{(n-1-(k-1))!(k-1)!} + \frac{(n-1)!}{(n-1-k)!k!}$$

$$= (n-1)!\left(\frac{1k}{(n-k)!(k-1)!k} + \frac{1n-k}{(n-k)(n-1-k)!k!}\right)$$

$$= \frac{(n-1)!}{(n-k)!k!}(k+(n-k))$$

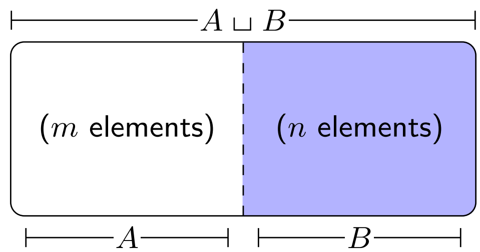
$$= \frac{(n-1)!}{(n-k)!k!} * n = \frac{n!}{(n-k)!k!} = \binom{n}{k}. \qquad \square$$

## Theorem (Vandermonde's identity)

*Let $m, n, r$ be nonnegative integers with $r \leqslant \min(m, n)$. Then*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Combinatorial proof sketch: count something in two ways. Let $A$ and $B$ be disjoint sets with $|A| = m$ and $|B| = n$.




Count the size-$r$ subsets of $A \sqcup B$ with $A$ and $B$ together, and then separately.

# See later:

Recall, Vandermonde's identity says that for $r \geq \min(m, n)$ we have $\boxed{\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}}$.

We showed this by counting size-$r$ subsets of $A \sqcup B$ in two ways, where $|A| = m$ and $|B| = n$.

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**Example of the RHS of proving Vandermonde's identity:** Let

$$A = \{a, b, c, d, e\}, \quad \text{and} \quad B = \{x, y, z\}, \quad \text{so that} \quad m = |A| = 5, \quad n = |B| = 3, \quad \text{and} \quad A \sqcup B = \{a, b, c, d, e, x, y, z\}.$$

Then the size-3 ($r = 3$) subsets can be broken up into 4 ($r + 1$) categories as follows:

| 3 elements from $A$, 0 from $B$ ($k = 0$) | 2 elements from $A$, 1 from $B$ ($k = 1$) | | | | 1 element from $A$, 2 from $B$ ($k = 2$) | | | | | 0 elements from $A$, 3 from $B$ ($k = 3$) |
|---|---|---|---|---|---|---|---|---|---|---|
| $\{a,b,c\}$ | $\{a,b,x\}$ | $\{b,c,x\}$ | $\{c,d,x\}$ | $\{d,e,x\}$ | $\{a,x,y\}$ | $\{b,x,y\}$ | $\{c,x,y\}$ | $\{d,x,y\}$ | $\{e,x,y\}$ | $\{x,y,z\}$ |
| $\{a,b,d\}$ | $\{a,b,y\}$ | $\{b,c,y\}$ | $\{c,d,y\}$ | $\{d,e,y\}$ | $\{a,x,z\}$ | $\{b,x,z\}$ | $\{c,x,z\}$ | $\{d,x,z\}$ | $\{e,x,z\}$ | |
| $\{a,b,e\}$ | $\{a,b,z\}$ | $\{b,c,z\}$ | $\{c,d,z\}$ | $\{d,e,z\}$ | $\{a,y,z\}$ | $\{b,y,z\}$ | $\{c,y,z\}$ | $\{d,y,z\}$ | $\{e,y,z\}$ | |
| $\{b,c,d\}$ | $\{a,c,x\}$ | $\{b,d,x\}$ | $\{c,e,x\}$ | | | | | | | |
| $\{b,c,e\}$ | $\{a,c,y\}$ | $\{b,d,y\}$ | $\{c,e,y\}$ | | | | | | | |
| $\{b,d,e\}$ | $\{a,c,z\}$ | $\{b,d,z\}$ | $\{c,e,z\}$ | | | | | | | |
| $\{c,d,e\}$ | $\{a,d,x\}$ | $\{b,e,x\}$ | | | | | | | | |
| $\vdots$ | $\{a,d,y\}$ | $\{b,e,y\}$ | | | | | | | | |
| | $\{a,d,z\}$ | $\{b,e,z\}$ | | | | | | | | |
| | $\{a,e,x\}$ | | | | | | | | | |
| | $\{a,e,y\}$ | | | | | | | | | |
| | $\{a,e,z\}$ | | | | | | | | | |
| there are $\binom{5}{3}\binom{3}{0}$ of these | there are $\binom{5}{2}\binom{3}{1}$ of these | | | | there are $\binom{5}{1}\binom{3}{2}$ of these | | | | | there are $\binom{5}{0}\binom{3}{3}$ of these |

So in total, there are

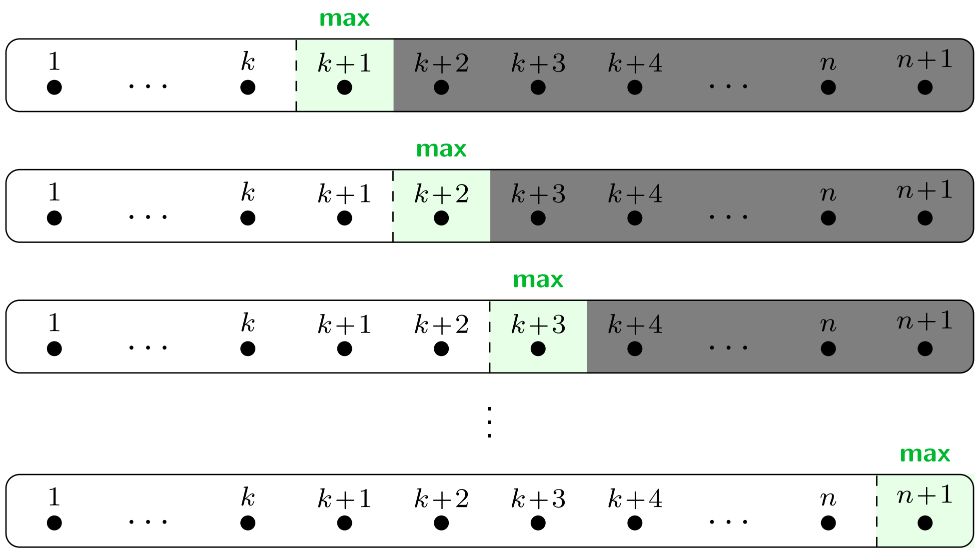
$$\binom{5}{3}\binom{3}{0} + \binom{5}{2}\binom{3}{1} + \binom{5}{1}\binom{3}{2} + \binom{5}{0}\binom{3}{3} = \sum_{k=0}^{3} \binom{5}{3-k}\binom{3}{k} = \binom{5+3}{3}$$

size-3 subsets of $A \sqcup B$.

Theorem. Let $n \geqslant k$ be nonnegative integers. Then

$$\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k}.$$

Combinatorial proof sketch: Put $n+1$ items in ascending order, and break into cases based on what the greatest chosen item is.



# 6.5: Generalized permutations and combinations

Caution: The book talks about *permutation* problems and *combination* problems **with or without repetition or replacement**. When the rest of the world says permutation or combination problem without clarification, they only mean without repetition/replacement!

**"Permutations with repetition":**

i.e. "ordered selection with replacement"

Permutation means that order matters.
Repetition means you can repeat objects.

Example questions:

1. How many 5-letter words are there? $\boxed{26^5}$
2. How many ways possible outcome are there of ten flips of a coin? $\boxed{2^{10}}$
3. How many ways ways can you answer a 20-question T or F quiz? $\boxed{3^{20}}$

Theorem. The number of ways to pick $n$ objects, in order, with possible repetition, from a set of $k$ objects is $\boxed{k^n}$.

# Permutations with indistinguishable objects.

Permutation means that order matters.
Indistinguishable means there are objects that can't be told apart.

Example questions:

1. How many 5-character passwords are there with 3 'A's, one 'B', and one 'C'?
   Place 3 A's: $\binom{5}{3}$, Place 1 B: $\binom{5-3}{1}$, Place 1 C: $\binom{5-3-1}{1}$.
   Total: $\binom{5}{3}\binom{2}{1}\binom{1}{1}$
2. How many anagrams are there of SUCCESS?
   Objects: 3 S's, 1 U, 2 C's, 1 E. Places: 7
   Place 3 S's: $\binom{7}{3}$, Place 1 U: $\binom{7-3}{1}$,
   Place 2 C's: $\binom{7-3-1}{2}$, Place 1 E: $\binom{7-3-1-2}{1}$.
   Total: $\binom{7}{3}\binom{4}{1}\binom{3}{2}\binom{1}{1}$

Solution strategy: Make a list of the objects and how many times they're used. Then place the objects one "type" at a time.

## Permutations with indistinguishable objects.

In general: the number of ways to place $n$ objects consisting of exactly

$$n_1 \text{ `}O_1\text{'s}, \quad n_2 \text{ `}O_2\text{'s}, \quad \ldots, \quad \text{and } n_k \text{ `}O_k\text{'s}$$

(so that $n = n_1 + \cdots + n_k$), in order is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_k}{n_k} \qquad \text{(since } n-(n_1+\cdots n_{k-1}) = n_k\text{)}.$$

Simplifying:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_k}{n_k}$$

$$= \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{n_k!}{n_k!0!}$$

$$= \boxed{\frac{n!}{n_1!n_2!n_3!\cdots n_k!}}.$$