Warmup

Recall

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Read notes for "Tips for being slick".

Then, without a calculator, compute the following values:

$$\binom{4}{2} \qquad \binom{5}{3}$$

$$\binom{8}{6} \qquad \binom{10}{3}$$

$$\binom{10}{9} \qquad \binom{15}{0}$$

$$\binom{200}{1} \qquad \binom{200}{200}$$

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$$\binom{5}{3} = 5 * 4/2 = \boxed{10}$$

$$\binom{8}{6} = 8 * 7/2 = \boxed{28}$$

$$\binom{10}{3} = \frac{10*9*8}{3*2} = \boxed{120}$$

$$\binom{10}{9} = \boxed{10}$$

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$$\begin{array}{rcccccl}
 & (x+y) & * & (x+y) & * & (x+y) & \\
= & x & * & x & * & x & \\
+ & x & * & x & * & y & \leftarrow \\
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$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

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Example: In the expansion of $(x+y)^{24}$, the coefficient of $x^{21}y^3$ is

$$\binom{21}{3} = \frac{21 * 20 * 19}{3 * 2 * 1}.$$

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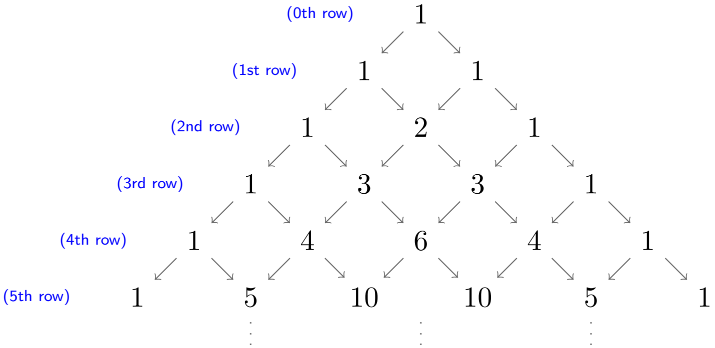
Recall Pascal's Triangle

Start and end each row with a 1. The ith row (starting with the 0th row) has $i+1$ entries. The middle entries are acquired by adding successive entries in the previous row.



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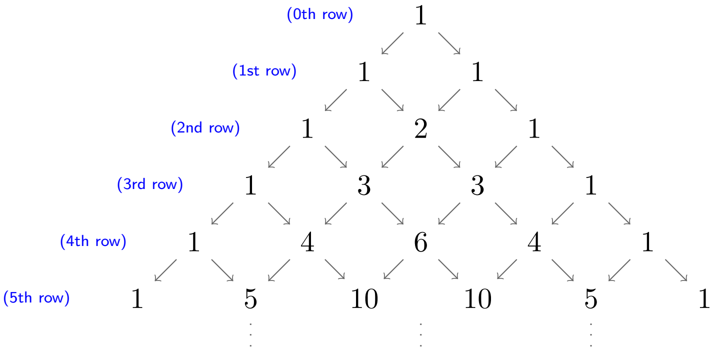

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Theorem (Pascal's identity)

Let n and k be positive integers with $n \geqslant k$. Then

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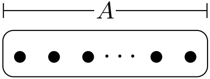

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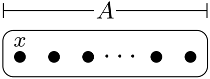

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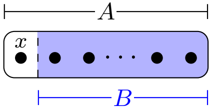

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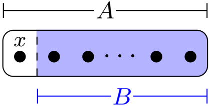

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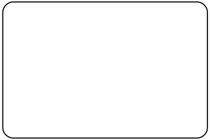

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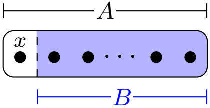

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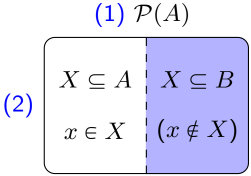

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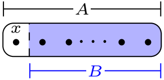

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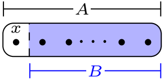

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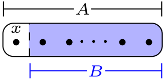

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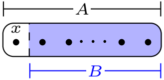

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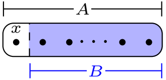

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$$\{X \subseteq A \mid x \notin X\} = \{X \subseteq B\} \quad \text{and} \quad |B| = n-1,$$
there are $\binom{n-1}{k}$ size-k subsets of A not containing x.
- Case 2: $x \in X$. Every size-k subset of A containing x looks like $X = \{x\} \cup Y$, where $Y \subseteq B$ and $|Y| = k-1$. There are $\binom{n-1}{k-1}$ of these.

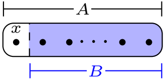
So, using the sum rule, there are $\binom{n-1}{k} + \binom{n-1}{k-1}$ subsets of A of size k.

Theorem (Pascal's identity). Let n and k be positive integers with $n \geqslant k$. Then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Combinatorial proof of Pascal's identity:

Let A be a set of size n. Fix $x \in A$ and let $B = A - \{x\}$.



Count size-k subsets of A in 2 ways:

LHS, All at once: There are $\binom{n}{k}$ subsets of A of size k.

RHS, Break into cases:

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Conclusion: Thus $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. □

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Theorem (Vandermonde's identity)

Let m, n, r be nonnegative integers with $r \leqslant \min(m, n)$. Then

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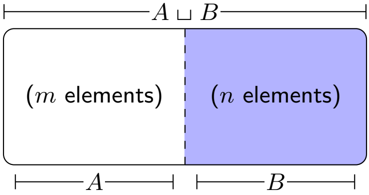

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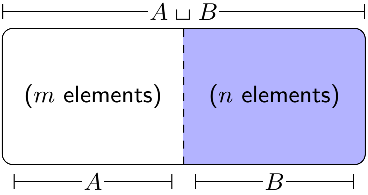

Count the size-r subsets of $A \sqcup B$ with A and B together, and then separately.

See later:

Recall, Vandermonde's identity says that for $r \geq \min(m,n)$ we have $\boxed{\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}}$.

We showed this by counting size-r subsets of $A \sqcup B$ in two ways, where $|A| = m$ and $|B| = n$.

Example of the RHS of proving Vandermonde's identity: Let

$$A = \{a,b,c,d,e\}, \quad \text{and} \quad B = \{x,y,z\}, \quad \text{so that} \quad m = |A| = 5, \quad n = |B| = 3, \quad \text{and} \quad A \sqcup B = \{a,b,c,d,e,x,y,z\}.$$

Then the size-3 ($r = 3$) subsets can be broken up into 4 ($r+1$) categories as follows:

3 elements from A, 0 from B ($k=0$)	2 elements from A, 1 from B ($k=1$)				1 element from A, 2 from B ($k=2$)					0 elements from A, 3 from B ($k=3$)
$\{a,b,c\}$	$\{a,b,x\}$	$\{b,c,x\}$	$\{c,d,x\}$	$\{d,e,x\}$	$\{a,x,y\}$	$\{b,x,y\}$	$\{c,x,y\}$	$\{d,x,y\}$	$\{e,x,y\}$	$\{x,y,z\}$
$\{a,b,d\}$	$\{a,b,y\}$	$\{b,c,y\}$	$\{c,d,y\}$	$\{d,e,y\}$	$\{a,x,z\}$	$\{b,x,z\}$	$\{c,x,z\}$	$\{d,x,z\}$	$\{e,x,z\}$	
$\{a,b,e\}$	$\{a,b,z\}$	$\{b,c,z\}$	$\{c,d,z\}$	$\{d,e,z\}$	$\{a,y,z\}$	$\{b,y,z\}$	$\{c,y,z\}$	$\{d,y,z\}$	$\{e,y,z\}$	
$\{b,c,d\}$	$\{a,c,x\}$	$\{b,d,x\}$	$\{c,e,x\}$							
$\{b,c,e\}$	$\{a,c,y\}$	$\{b,d,y\}$	$\{c,e,y\}$							
$\{b,d,e\}$	$\{a,c,z\}$	$\{b,d,z\}$	$\{c,e,z\}$							
$\{c,d,e\}$	$\{a,d,x\}$	$\{b,e,x\}$								
$\vdots$	$\{a,d,y\}$	$\{b,e,y\}$								
	$\{a,d,z\}$	$\{b,e,z\}$								
	$\{a,e,x\}$									
	$\{a,e,y\}$									
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there are $\binom{5}{3}\binom{3}{0}$ of these	there are $\binom{5}{2}\binom{3}{1}$ of these				there are $\binom{5}{1}\binom{3}{2}$ of these					there are $\binom{5}{0}\binom{3}{3}$ of these

So in total, there are

$$\binom{5}{3}\binom{3}{0} + \binom{5}{2}\binom{3}{1} + \binom{5}{1}\binom{3}{2} + \binom{5}{0}\binom{3}{3} = \sum_{k=0}^{3} \binom{5}{3-k}\binom{3}{k} = \binom{5+3}{3}$$

size-3 subsets of $A \sqcup B$.

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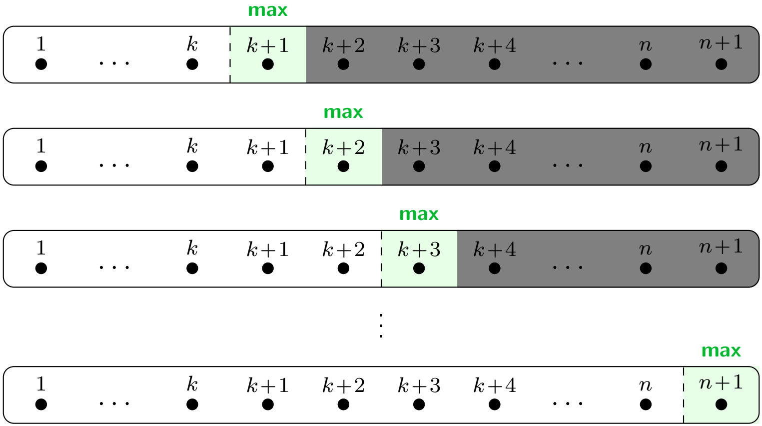

6.5: Generalized permutations and combinations

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1. How many 5-letter words are there? $\boxed{26^5}$
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Theorem. The number of ways to pick n objects, in order, with possible repetition, from a set of k objects is $\boxed{k^n}$.

Permutations with indistinguishable objects.

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Indistinguishable means there are objects that can't be told apart.

Example questions:

1. How many 5-character passwords are there with 3 'A's, one 'B', and one 'C'?

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In general: the number of ways to place n objects consisting of exactly

$$n_1 \text{ `}O_1\text{'s}, \quad n_2 \text{ `}O_2\text{'s}, \quad \ldots, \quad \text{and } n_k \text{ `}O_k\text{'s}$$

(so that $n = n_1 + \cdots + n_k$), in order is

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