Math 365 – Monday 2/25/19
Section 6.3 & 6.2: Permutations, combinations, and pigeonhole principle

Exercise 22.

(a) Consider the set $\{a, b, c\}$. For each of the following, (A) list the objects described, (B) give a formula that tells you how many you should have listed, and (C) verify that the formula and the list agree.
  (i) Permutations of $\{a, b, c\}$.
  (ii) Two-permutations of $\{a, b, c\}$.
  (iii) Size-two subsets of $\{a, b, c\}$.

(b) For each of the following, classify the problem as a permutation or a combination problem or neither, and give an answer using an unsimplified formula. (Answers should look, for example, like $5 * 4$ or $5!/2!$ instead of $P(5, 4)$ or 20.)
  (i) In how many different orders can five runners finish a race if no ties are allowed?
  (ii) How many strings of 1's and 0's of length seven have exactly three 1's?
  (iii) How many strings of 1's and 0's of length seven have three or fewer 1's?
  (iv) How many three-digit numbers are there with no 1's? (a three-digit number is something like 144 or 009 or 053)
  (v) How many three-digit numbers are there with no digits repeated?

(c) For each of the following, provide your answers in an unsimplified form, and justify.
  (i) A six-sided dice is rolled 5 times. How many ways could it turn out that a value greater than 4 (i.e. 5 or 6) is rolled exactly twice? *(Hint: first pick which rolls are from $\{5, 6\}$ (this implies which rolls are from $\{1, 2, 3, 4\}$ for free), and then pick the values for the $\{5, 6\}$-valued rolls, and finally pick the values for the other rolls (the $\{1, 2, 3, 4\}$-valued rolls).)*
  (ii) If 10 men and 10 women show up for one team of an intramural basketball game, how many ways can you pick 5 people to play for one team if there must be at least one person of each gender on the team?
  (iii) How many ways are there for 5 women and 2 men to stand in line? Now how many ways are there for them to stand in line if the two men don't stand next to each other? (The men and the women are distinct individuals.)

(d) For each of the following identities, (A) explain in words why it makes sense given what it represents, and then (B) verify it algebraically using the formulas for permutation or combination. *(For example, an answer for (A) might start out looking like "$P(n, 1)$ means. . . ", and an answer for part (B) should look like a calculation that starts with "$P(n, 1) = \ldots$ ".)*
  (i) $P(n, 1) = n$
  (ii) $P(n, 0) = 1$
  (iii) $P(n, k + 1) = P(n, k) * (n - k)$
  (iv) $\binom{n}{1} = n$
  (v) $\binom{n}{n} = 1$
  (vi) $\binom{n}{0} = 1$
  (vii) $\binom{n}{k} = \binom{n}{n-k}$

Exercise 23. For each of the following, be sure to include how the pigeonhole principle or its generalized version apply in your justifications, or why neither of them do.

(1) The lights have gone out and you're digging through an unorganized sock drawer filled with unmatched black socks and brown socks (otherwise roughly identical).
   (a) If you're pulling them out at random, how many socks do you need to take out to ensure you have a matching pair if there are 10 of each kind of sock? How about if there are 20 of each? 100 of each?
   (b) Again pulling at random, how many socks do you need to take out to ensure you have a matching brown pair if there are 10 of each kind of sock? How about if there are 20 of each? 100 of each?

(2) Explain why, out of any set of four integers, at least two have the same remainder when divided by 3.

(3) A recent estimate showed that the US and Canada together (which share the country code +1) have approximately 134,000,000 phone lines in use. What is the minimum number of area codes needed to make that possible?

(4) Let $f : A \to B$ be a function between finite sets such that $|A| > |B|$. Explain why f cannot possibly be injective. (Consider the sizes of the preimages $\{f^{-1}(b) \mid b \in B\}$.)

(5) Explain why, in any sequence of n consecutive integers, at least one of them must be divisible by n. (Start with, say, $n = 4$ as an example.)

Common errors on the quizzes

1. Define "cardinality": *Two sets X and Y have the same cardinality if...*

[Hint: "...they have the same size" is not the right answer.]

X and Y have the same number of elements

Problem: what about $\mathbb{R}$ versus $\mathbb{Z}$? Or $\mathbb{Z}$ versus $\mathbb{Z}_{\geqslant 1}$?

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1. Define "cardinality": *Two sets X and Y have the same cardinality if...*

[Hint: "...they have the same size" is not the right answer.]

X and Y are bijective

Problem: We defined "bijective" for functions, not sets. Like "these flowers smell bijective".

Good:

1. Define "cardinality": *Two sets X and Y have the same cardinality if...*

[Hint: "...they have the same size" is not the right answer.]

X and Y are in bijection with eachother.

1. Define "cardinality": *Two sets X and Y have the same cardinality if...*

[Hint: "...they have the same size" is not the right answer.]

there is a bijective function $f: X \to Y$.

Bad example.

Claim: $-1 = 1$

Non-proof.

If $-1 = 1$, then

$$(-1)^2 = (1)^2, \qquad \text{so that} \qquad 1 = 1,$$

which is true. □

What went wrong:

We proved that

"$-1 = 1$ implies $1 = 1$",

which *is* (strangely enough) true. A false statement can imply a true statement.

We **did not** show that $-1 = 1$.

"Outline a proof by induction that $\sum_{i=1}^{n} = n(n+1)/2$.

Inductive step

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

$$1+2+\cdots+n+n+1 = \frac{(n+1)(n+2)}{2}$$
$$-(n+1) \qquad\qquad -(n+1)$$

$\times 2$:

$$2(1+2+\cdots+n) = (n+1)(n+2)-2(n+1)$$

$$2\sum_{i=1}^{n} i = n^2+3n+2-n-2$$

$$2\left(\frac{(n+1)n}{2}\right) = n^2+2n$$

$$(n+1)n = n^2+2n \quad ✓$$

Problem: Manipulating formulas from "what you want" to "something true" isn't very reliable.

"Outline a proof by induction that $\sum_{i=1}^{n} = n(n+1)/2$.

Good:

- Inductive step: Fix $n \geq 1$, and assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (for that n).

Then
$$\sum_{i=1}^{n+1} i = \underbrace{1+2+\cdots+n}_{\sum_{i=1}^{n} i} + (n+1)$$
$$= \frac{n(n+1)}{2} + (n+1) \quad \text{(by the induction hypothesis)}$$
$$= \frac{n^2+n+2n+2}{2} = \frac{(n+1)(n+2)}{2}.$$

"Outline a proof by induction that $\sum_{i=1}^{n} = n(n+1)/2$.

One more problem:

$P(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ versus $P(n) : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Problem: $P(n)$ is a *statement*, like "avocados are green", and can't equal anything.

On the other hand, $n(n+1)/2$ is an *expression*, which is basically a noun, like "avocados".

Last time: Counting rules.

Product: If a procedure can be broken into a sequence of two tasks, and there are n_1 ways to do the first task, and for each of these ways of doing the first task, there are n_2 ways to doing the second task, then there are $n_1 n_2$ total distinct outcomes.

Sum: If a procedure can be done either in one of n_1 ways or in one of n_2 ways, where there is no overlap in the n_1 and n_2 ways, then there are $n_1 + n_2$ total distinct outcomes.

Subtraction/Inclusion-exclusion: If a procedure can be done either in one of n_1 ways or in one of n_2 ways, but there are n_3 overlapping outcomes, then there are $n_1 + n_2 - n_3$ total distinct outcomes.

Division: If a procedure can be done in n ways, but that procedure produces each outcome in d different ways, then there are actually n/d distinct outcomes.

Remember: Every rule depends on making up a "procedure" for counting, and then applying the rules according to that procedure!!
(Take it from this expert: *Never* just plug stuff into a formula – make up a story for counting things one step at a time, and *then* try to count.)

Section 6.3: Permutations and combinations

The choose function is

$$C(n,k) = \binom{n}{k} = \#\{ \text{ ways to choose } k \text{ objects from } n \ \},$$

read "n choose k".
Now that we have some counting skills, we can build a formula using the product and division rules.
Let's start with an example...

$$C(n,k) = \binom{n}{k} = \#\{ \text{ ways to choose } k \text{ objects from } n \ \}$$

Let's start with an example: $n = 7$, $k = 3$.

Step 1: How many ways are there to select 3 things from 7 in order (no replacement)?

Ans: $7 * 6 * 5$.
Rewriting, notice that

$$7! = (7 * 6 * 5) * \underbrace{(4 * 3 * 2 * 1)}_{(7-3)!=4!},$$

so

$$7 * 6 * 5 = \frac{7!}{(7-3)!}.$$

$$C(n,k) = \binom{n}{k} = \#\{ \text{ ways to choose } k \text{ objects from } n \ \}$$

Let's start with an example: $n = 6$, $k = 3$.
Step 1: There are $7!/(7-3)!$ ways to select 3 things from 7 in order.
Step 2: How many ways are there to put the 3 things in order? (How many permutations are there of 3 things?)

Ans: $3 * 2 * 1 = 3!$.

Step 3: Use the division rule to combine step 1 and step 2. Namely, the the number of ways to choose 3 things without order from 7 is

$$\frac{\text{step 1}}{\text{step 2}} = \frac{7!/(7-3)!}{3!} = \frac{7!}{(7-3)!3!}.$$

So $\boxed{C(7,3) = \frac{7!}{(7-3)!3!}}$.

$$C(n,k) = \binom{n}{k} = \#\{\text{ ways to choose } k \text{ objects from } n \text{ }\}$$

Now for general $n \geqslant k \geqslant 1$:

Step 1: How many ways are there to select k things from n in order (no replacement)?

Ans: $n*(n-1)*(n-2)\cdots(n-(k-1))$.
Rewriting, notice that

$$n! = \big(n*(n-1)\cdots(n-(k-1))\big)*\underbrace{\big((n-k)*(n-k-1)*\cdots 2*1\big)}_{(n-k)!},$$

so

$$n*(n-1)*(n-2)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}.$$

$$\binom{n}{k} = \#\{\text{ ways to choose } k \text{ objects from } n \text{ }\}$$

Now for general $n \geqslant k \geqslant 1$:
Step 1: There are $n!/(n-k)!$ ways to select k things from n in order.
Step 2: How many ways are there to put the k things in order? (How many permutations are there of k things?)

Ans: $k*(k-1)*(k-2)*\cdots*2*1 = k!$.

Step 3: Use the division rule to combine step 1 and step 2. Namely, the number of ways to choose k things without order from n is

$$\frac{\text{step 1}}{\text{step 2}} = \frac{n!/(n-k)!}{k!} = \frac{n!}{(n-k)!k!}.$$

So $\boxed{C(n,k) = \frac{n!}{(n-k)!k!}}$.

$C(n,k) = \binom{n}{k} = \#\{$ ways to choose k objects from n $\} = \frac{n!}{(n-k)!k!}$

A k-permutation of n objects is a choice of k things from n **in order**. The permutation function is

$$P(n,k) = \#\{ \text{ ways to select } k \text{ objects from } n \text{ in order}\}.$$

As we saw,

$$P(n,k) = \frac{n!}{(n-k)!}.$$

A permutation is an n-permutation of n objects.

We call counting problems that call for unordered selection "combination problems".
We call counting problems that call for ordered selection "permutation problems".

$C(n,k) = \binom{n}{k} = \#\{$ ways to choose k objects from n $\} = \frac{n!}{(n-k)!k!}$

$P(n,k) = \#\{$ ways to select k objects from n in order $\} = \frac{n!}{(n-k)!}$

Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (Permutation, $n = 20$, $k = 3$)

$\boxed{P(20,3) = 20 * 19 * 18.}$

(2) How many ways can one pick a committee of 3 from a club of 20 people? (Combination, $n = 20$, $k = 3$)

$\boxed{C(20,3) = 20 * 19 * 18/(3 * 2 * 1).}$

(3) A coin is flipped 5 times. How many ways could it turn out that heads comes up exactly 3 times?
Ans: When I choose the 3 times that the heads come up, it doesn't matter what order I choose them, just which slots I pick. (Combination, $n = 5$, $k = 3$)

$\boxed{C(5,3) = 5 * 4 * 3/(3 * 2 * 1).}$

$C(n,k) = \binom{n}{k} = \#\{$ ways to choose k objects from n $\} = \frac{n!}{(n-k)!k!}$
$P(n,k) = \#\{$ ways to select k objects from n in order $\} = \frac{n!}{(n-k)!}$

Some examples:

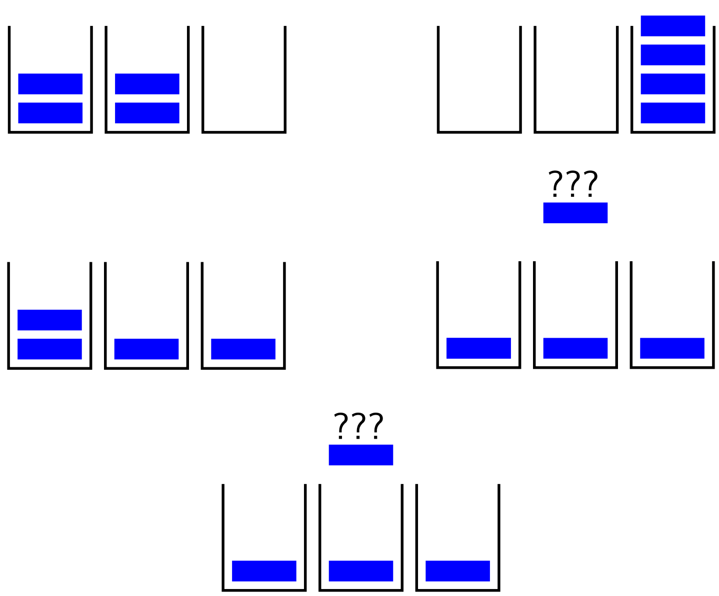
(4) True or false problem from the in-class exercises last time: How many ways can a student answer a True or False quiz with 10 questions if they may or may not leave a problem blank?
(Counting in two different ways gives the identity

$$3^{10} = \sum_{i=0}^{10} \binom{10}{i} 2^{10-i}.)$$

You try: Exercise 22

Pigeonhole principle - §6.2

The Pigeonhole Principle says that if $k+1$ objects are placed into k boxes, then at least one box contains two or more objects.



Proof (by contradiction).

Suppose every box contains at most one object. Then there are at

Pigeonhole Principle: "if $k+1$ objects are placed into k boxes, then at least one box contains two or more objects."

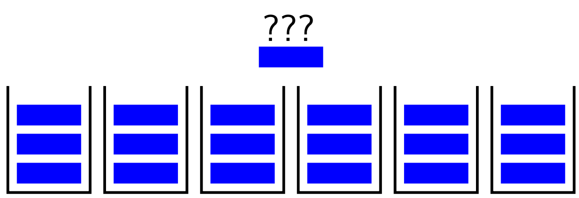
Ex: In any group of 367 people, at least two of those people have the same birthday.

NON-Ex: In a group of 367 people, it is not guaranteed that at least two people were born on a *specific* day. For example, it is not guaranteed that at least two people were born on January 1st. (It is not guaranteed that *any* of them were born on January 1st!)

Ex: In any set of 27 english words, at least two start with the same letter; at least two end with the same letter.

NON-Ex: In a set of twenty seven english words, it is not guaranteed that at least two start with a *specific* letter. For example, it is not guaranteed that at least two start with 'z' (say, the twenty seven distinct words in this example).

The Generalized Pigeonhole Principle says that if N objects are placed into k boxes, then at least one box contains $\lceil N/k \rceil$ objects.



Proof (by contradiction).

Note that

$$\lceil N/k \rceil < (N/k) + 1$$

(the ceiling function rounds up, which increases a value by less than 1). So, multiplying both sides by k, we get

$$k\lceil N/k \rceil < k((N/k) + 1) = N + k.$$

Now suppose every box contains at most $\lceil N/k \rceil - 1$ objects (note $\lceil N/k \rceil \geqslant 1$). Then there are at most

$$\#\{\text{ boxes }\} * (\text{ max } \# \text{ objects per box }) = k * (\lceil N/k \rceil - 1)$$
$$= k\lceil N/k \rceil - k < (N + k) - k = N.$$

This is a contradiction. So least one box contains $\lceil N/k \rceil$ or more objects. □

The Generalized Pigeonhole Principle says that if N objects are placed into k boxes, then at least one box contains $\lceil N/k \rceil$ objects.

Example: What is the minimum number of students required in a class to ensure that at least ten people will receive the same grade (if the grade options are just A,B,C,D,F)?

Answer:
Here, the grades are the “boxes”, of which there are 5. ($k = 5$).
Need at least one of the boxes to have at least ten students.
($\lceil N/5 \rceil \geqslant 10$)
But

$$\lceil N/5 \rceil \geqslant 10 \quad \text{exactly when} \quad N/5 > 9.$$

So the question comes down to finding the least integer N such that $N > 5 * 9$: $\boxed{N = 5 * 9 + 1}$

The Generalized Pigeonhole Principle says that if N objects are placed into k boxes, then at least one box contains $\lceil N/k \rceil$ objects.

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit appear?

Answer:
Here, the suits are the “boxes”, of which there are 4. ($k = 4$)
Need at least one of the boxes to have at least three cards.
($\lceil N/4 \rceil \geqslant 3$)
But

$$\lceil N/4 \rceil \geqslant 3 \quad \text{exactly when} \quad N/4 > 2.$$

So the question comes down to finding the least integer N such that $N > 4 * 2$: $\boxed{N = 4 * 2 + 1}$

The Generalized Pigeonhole Principle says that if N objects are placed into k boxes, then at least one box contains $\lceil N/k \rceil$ objects.

NON-Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three hearts appear?

Answer: Since there are 39 cards that are not hearts, we would need to pull at least $39 + 3 = 42$ cards to ensure at least three hearts.

Example: What is the least number of area codes needed to ensure the availability of at least 25 million distinct phone numbers? (A valid phone number is a sequence of 10 digits, where the first three are the area code, and the 1st and 4th are not 1's or 0's.)
Ans: Here, the seven digit phone numbers are the "boxes", of which there are $8 * 10^6 = 8$ million. ($k = 8$ million)
So if we have 25 million phone numbers, at least

$$\lceil 25\text{mil}/8\text{mil} \rceil = 4$$

will have the same 7-digit phone number. Therefore we need $\boxed{4}$ area codes.
You try: Exercise 23