Math 365 – Wednesday 2/20/19 – Section 6.1: Basic counting

Exercise 19. For each of the following, use some combination of the sum and product rules to find your answer. Give an un-simplified numerical answer (i.e. give your answer as, say, $5 * 4$ instead of 20), and explain it, saying which rule(s) you're using when.

(a) A particular kind of shirt comes in two different cuts - male and female, each in three color choices and five sizes. How many different choices are made available?
(b) On a ten-question true-or-false quiz, how many different ways can a student fill out the quiz if they definitely answer all of the questions? How many ways if they might leave questions blank?
(c) How many 3-letter words (these don't have to be real words, just strings of letters) are there?
(d) How many 3-letter words are there that end in a vowel?
(e) How many 3-letter words are there that have no repeated characters?
(f) How many 3-letter words are there that have the property that if they start in a vowel then they don't end in a vowel? (You'll want to break this into disjoint cases).
(g) How many 2-letter passwords are there that are made up of upper and/or lower case letters?
(h) How many 2-letter passwords are there that are made up of upper and/or lower case letters, but where at least one of the letters is upper-case? (Again, you'll want to break this into disjoint cases).

To check your answers: (a) 30; (b) 1024; 59,049; (c) 17,576; (d) 3380; (e) 15,600; (f) 16,926; (g) 2704; (h) 2028.

Exercise 20. For each of the following, use some combination of the sum, product, inclusion-exclusion, and division rules to find your answer. Give an un-simplified numerical answer, and explain it, saying which rule(s) you're using when. (Note, there are 26 letters and 5 vowels.)

(a) How many strings of three letters are there that satisfy the following:
    (i) that contain exactly one vowel?
    (ii) that contain exactly 2 vowels?
    (iii) that contain at least 1 vowel?
(b) How many 3-card hands are there from a 52-card deck, which...
    (i) have no other restrictions?
    (ii) are all hearts?
    (iii) are all the same suit?
    (iv) form a straight (like 2,3,4 in possibly mixed suits. Note that Ace, 2, 3 and Queen, King, Ace are both straights.)
    (v) are not all the same suit? (use two of your previous answers)
(c) How many ways are there to seat 6 people at a round table with 6 chairs, if you're only paying attention to who is sitting next to whom?
(d) How many positive integers (strictly) less than 100 are there that are divisible by 2 and/or 3?

To check your answers: (a) 6615; 1575; 8315; (b) 22,100; 286; 1144; 768; 20,956; (c) 60; (d) 66.

Exercise 21.

(a) How many positive integers less than 1000
   (i) are divisible by 7?
   (ii) are divisible by 7 but not by 11?
   (iii) are divisible by both 7 and 11?
   (iv) are divisible by either 7 or 11?
   (v) are divisible by exactly one of 7 and 11?
   (vi) are divisible by neither 7 nor 11?
   (vii) have distinct digits?
   (viii) have distinct digits and are even?

(b) How many strings of three decimal digits
   (i) do not contain the same digit three times?
   (ii) begin with an odd digit?
   (iii) have exactly two digits that are 4s?

(c) How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

(d) Suppose that at some future time every telephone in the world is assigned a number that contains a country code that is 1 to 3 digits long, that is, of the form X, XX, or XXX, followed by a 10-digit telephone number of the form NXX-NXX-XXXX (as described in Example 8 in the book). How many different telephone numbers would be available worldwide under this numbering plan?

(e) How many injective functions are there from $\{a, b, c\}$ to $\{1, 2, 3, 4\}$?

(f) How many surjective functions are there from $\{a, b, c, d\}$ to $\{1, 2, 3\}$?
[Hint: If $f : \{a, b, c, d\} \to \{1, 2, 3\}$ is surjective, then exactly one of $1, 2$, or 3 has a preimage of size 2. First choose which of those three elements has the larger preimage, then pick it's preimage, and then assign the other two preimages.]

Section 6.1: The basics of counting

Basic counting rules:

The product rule: Suppose a procedure can be broken into a sequence of two tasks.

If there are n_1 ways to do the first task, and

for each way of doing the first task, there are n_2 ways to doing the second task,

then there are $n_1 n_2$ ways in total to do the procedure.

Example: Chairs in an auditorium are labeled with a letter followed by a positive integer not exceeding 100. What is the maximum number of chairs that can be labeled differently? $\boxed{26 * 100}$

Example: If forty people come to class, each wearing a pair of shoes, how many shoes are there in the room? $\boxed{40 * 2}$

Example: If there are 20 people in a competition for which there is a first and second prize, how many possible outcomes are there? $\boxed{20 * 19}$

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Example: How many functions are there from a set of size three to a set of size 5? $\boxed{5 * 5 * 5}$

How many injective functions from a set of size three to a set of size 5? $\boxed{5 * 4 * 3}$

NON Example: How many ways are there to choose a pair of cards from a deck of 52 cards? (There is no first and second tasks - the two cards are drawn at the same time.)

Sets: In set language, the product rule is the same as the rule for computing the size of a cartesian product of finite sets $A_1, \ldots, A_n$:

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1||A_2| \cdots |A_n|.$$

Section 6.1: The basics of counting

Basic counting rules:

The sum rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where there is no overlap in the n_1 and n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: If a student council member is going to be chosen from the first and second year student body, where there are 1503 first-year students and 1475 second-year students, how many possible candidates are there? $\boxed{1503 + 1475}$

NON example: An internship will be available to math majors, of which there are 120, and physics majors, of which there are 200. How many potential applicants might there be for the internship? (There may be overlap amongst math majors and physics majors.)

Sets: In set language, the sum rule is the same as the fact that for some pairwise disjoint finite sets $A_1, \ldots, A_n$, i.e. $A_i \cap A_j = \varnothing$ for all $i \neq j$, we have

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

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The sum rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where there is no overlap in the n_1 and n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example of combining sum and product rules:

Say passwords for a site are required to be 6-8 characters long, using upper and/or lower case letters and/or numbers. How many possible passwords are there?

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Answer: First we use the sum rule to decide how many possibilities for each character: $26 + 26 + 10 = 62$. (sum)

Next, break the problem into three cases: A password is either 6 or 7 or 8 letters long.

6 characters: There are $(26 + 26 + 10)^6 = 62^6$ possibilities. (prod)

7 characters: There are $(26 + 26 + 10)^7 = 62^7$ possibilities. (prod)

8 characters: There are $(26 + 26 + 10)^8 = 62^8$ possibilities. (prod)

So in total, there are $62^6 + 62^7 + 62^8$ possible passwords. (sum)

You try: In-class exercise 19.

More rules

The subtraction rule: If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Example: An internship will be available to math majors, of which there are 120, and physics majors, of which there are 200, but there are 15 students double-majoring in math and physics. How many potential applicants might there be for the internship?

$\boxed{120 + 200 - 15}$

More rules

The subtraction rule: If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Example: How many two-character passwords are there that are made up of upper and lower case letters and numbers, but where at least one of the characters is a number?

Answer: There are $10 * 62$ passwords where the first character is a number (prod), $62 * 10$ passwords where the second character is a number (prod), and $10 * 10$ passwords where both characters are numbers (prod). So there are

$$10 * 62 + 62 * 10 - 10 * 10$$

valid passwords in total (sub). //

In set theory language, we call this the inclusion-exclusion principle:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Sometimes we say we "double counted", and have to fix it.

More rules

The division rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, but for each way w, d of the ways have the same outcome w.

Example: How many ways are there to choose a pair of cards from a deck of 52 cards?
Answer: Using the product rule, we can choose the cards in $52 * 51$ ways if we draw them one at a time in order ($n = 52 * 51$). But for each pair $\{\text{card}_A, \text{card}_B\}$, there are two ways to get that pair in this way:

card_A first, card_B second, or
card_B first and card_A second.

(So $d = 2$.) Answer: $\boxed{52 * 51/2}$

More rules

The division rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, but for each way w, d of the ways have the same outcome w.

Example: How many ways can you choose a committee of 3 from 10 people?
Answer: Using the product rule, we can choose the people in $10 * 9 * 8$ ways if we choose them one at a time in order ($n = 10 * 9 * 8$). But for each committee

$$\{\text{member}_A, \text{member}_B, \text{member}_C\},$$

there are $3 * 2 * 1$ ways to get that committee in this way:

any 3 of them could have been chosen first,
then any 2 of the remaining second,
and whoever is left last.

(So $d = 3!$.) Answer: $\boxed{(10 * 9 * 8)/(3!)}$

You try: In-class exercise 20.