## Mathematical Induction

Sorites paradox: If $1,000,000$ grains of sand forms a "heap of sand", and removing one grain from a heap leaves it a heap, then a single grain of sand (or even no grains) still forms a heap.

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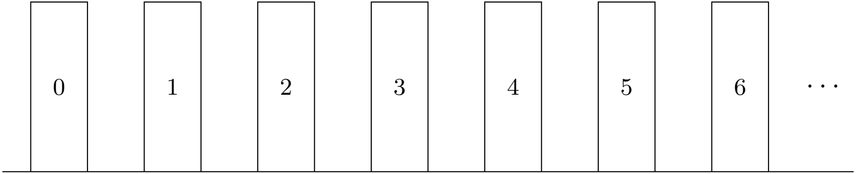
For example, say you have an infinite row of dominoes, labeled $0, 1, 2, \ldots$:



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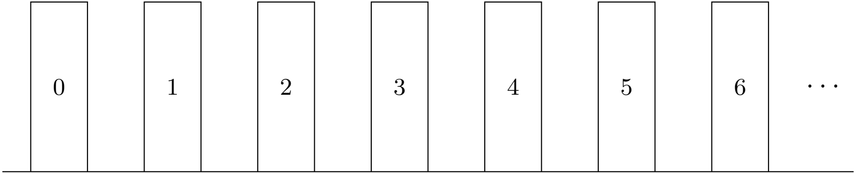
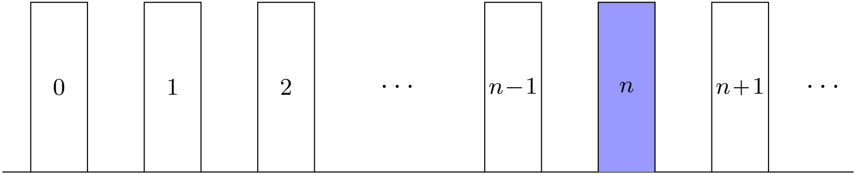



Let $P(n)$ be the statement

"I can knock the $n$th domino over".



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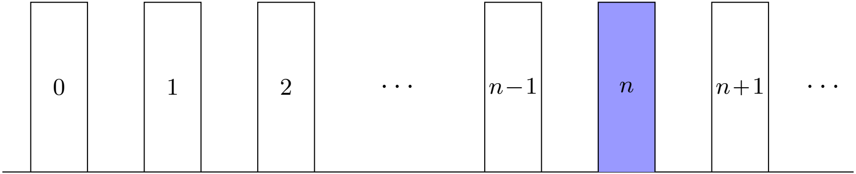
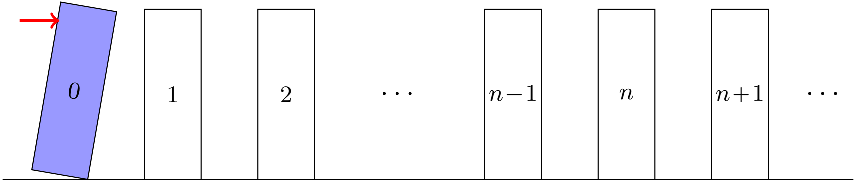



If you can start by bumping the 0th domino over, that's showing that $P(0)$ is true:



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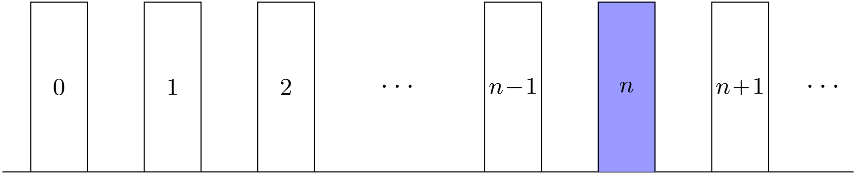
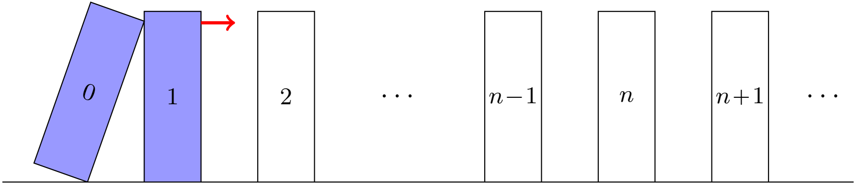



Then, if you can show that the 0th domino knocking into the 1st domino with then knock #1 over, you'll show that $P(1)$ is true:



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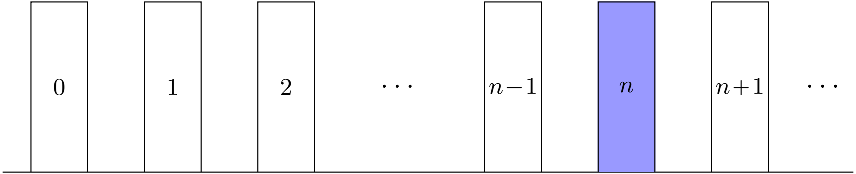


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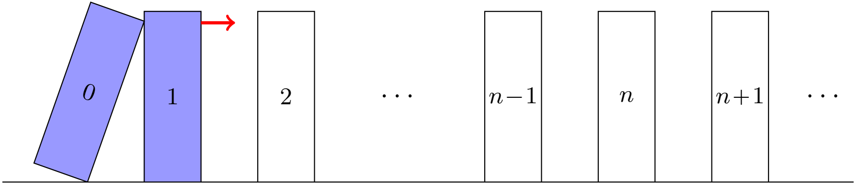


In math: You can show that $P(1)$ is true by proving (a) $P(0)$ is true, and (b) that $P(0)$ implies $P(1)$.

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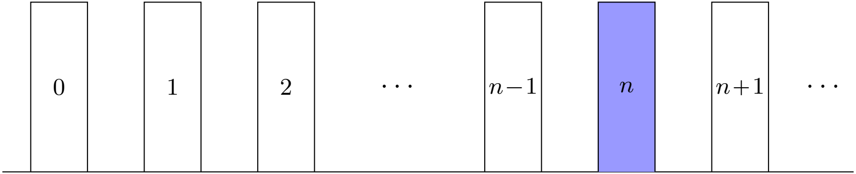


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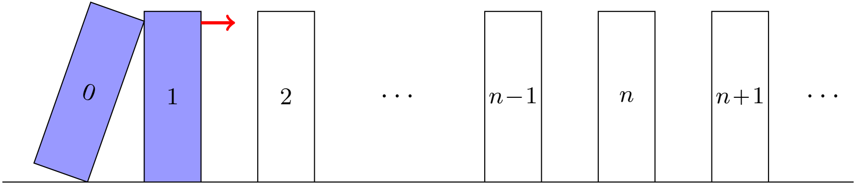


In math: You can show that $P(1)$ is true by proving (a) $P(0)$ is true, and (b) that $P(0)$ implies $P(1)$.

Idea: $P(1)$ will imply $P(2)$, which will imply $P(3)$, and so on...

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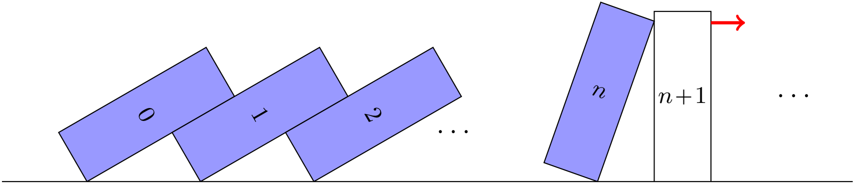
To show that $P(k)$ holds in general, you show that

(a) $P(0)$ is true, and then

(b) for any $n$, if $P(n)$ is true, then that implies $P(n+1)$ is also true. (If the $n$th domino falls, then so will the $(n+1)$th)



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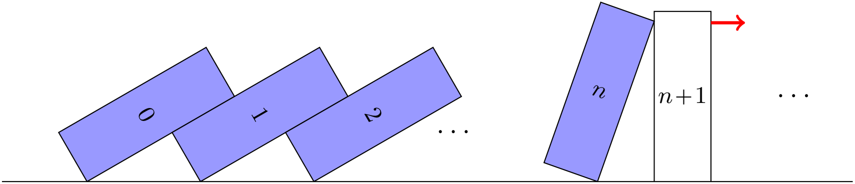
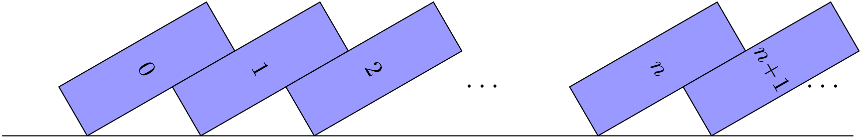



Then by letting the dominos fall one after the other, eventually each domino will fall (no particular domino will be left standing, given enough time):



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Thus, by induction, I can knock down the $k$th domino for any $k \in \mathbb{Z}_{\geqslant 0}$. $\square$

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**Theorem**: for any $k \in \mathbb{Z}_{\geqslant 0}$, I can knock down the $k$th domino.

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**Conclusion:** So since $P(1)$ is true, and $P(n)$ implies $P(n+1)$, we have $P(k)$ is true for all $k = 1, 2, 3, \ldots$. □

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Proof by induction (final draft). For $n = 1$, we have

$$\sum_{i=1}^{1} i = 1 = \frac{1 * 2}{2},$$

as desired. Now fix $n \geqslant 1$ and assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (for that value of $n$). Then

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \underbrace{1 + 2 + \cdots + n}_{\sum_{i=1}^{n} i} + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \quad \text{(by the inductive hypothesis)} \\ &= \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}.\end{aligned}$$

Thus, the claim holds for all $n \geqslant 1$ by induction. □

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**Inductive step:** (Assume $P(n)$ and show $P(n+1)$)

Fix $n \geqslant 0$ and assume $n < 2^n$ (this is the IH). Then since $n \geqslant 0$,

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Of course, we could have shown this directly!

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Example: Show that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Proof by induction (final draft). For $n = 0$, we have $A = \varnothing$, and so $\mathcal{P}(A) = \{\varnothing\}$. Thus

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as desired. Now fix $n \geqslant 0$ and assume for any size-$n$ set $A$, we have $|\mathcal{P}(A)| = 2^n$. Let $B$ be a set of size $n + 1$, and let $b \in B$. Let $A = B - \{b\}$, so that

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Outlining your proof:

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You try: Exercise 17.