Sequences

A *sequence* is a function a from a subset of the set of integers (usually $\mathbb{Z}_{\geqslant 0}$ or $\mathbb{Z}_{>0}$) to a set S,

$$a : \mathbb{Z}_{\geqslant 0} \to S \qquad \text{or} \qquad a : \mathbb{Z}_{>0} \to S.$$

We write $a_n = a(n)$, and call a_n the *nth term of the sequence*.

Example

The sequence defined by the function

$$a : \mathbb{Z}_{>0} \to \mathbb{Q} \qquad \text{defined by} \qquad n \mapsto 1/n$$

is the sequence

$$1, 1/2, 1/3, 1/4, \ldots.$$

We write $a_n = 1/n$.

We can also write such a sequence like

$$\{a(n)\}_{n=1,2,\ldots} \qquad \text{or} \qquad \{a(n)\}_{n\in\mathbb{Z}_{>0}}.$$

For example, the sequence above is $\{1/n\}_{n\in\mathbb{Z}_{>0}}$.

Some different kinds of sequences

A *geometric sequence* (or *progression*) is a sequence of the form

$$c, cr, cr^2, cr^3, \ldots, \qquad \text{i.e.} \qquad a : \mathbb{Z}_{\geqslant 0} \to S \quad \text{by} \quad n \mapsto cr^n,$$

for some constants c and r. (This is a discrete version of the exponential function $f(x) = cr^x$.)

An *arithmetic progression* is a sequence of the form

$$b,\ b+m,\ b+2m,\ b+3m, \ldots,$$

$$\text{i.e.} \qquad a : \mathbb{Z}_{\geqslant 0} \to S \quad \text{by} \quad n \mapsto b + mn,$$

for some constants b and m. (This is a discrete version of the linear function $f(x) = b + mx$.)

Notice, with a geometric sequence, the *ratio is constant*:

if $a_n = cr^n$, then $a_n/a_{n-1} = r$ for all n.

And with an arithmetic sequence the *difference is constant*:

if $a_n = b + mn$, then $a_n - a_{n-1} = m$ for all n.

(This is how we test to see if a sequence is geometric or arithmetic!)

Recurrence relations

A *recurrence relation* for a sequence is an equation that expresses a_n in terms of one of more of the previous terms of the sequence. For example:

$$a_n = a_{n-1} * 2;$$
$$a_n = a_{n-2} + 1;$$
$$a_n = a_{n-1} + a_{n-2}.$$

A sequence is called a *solution* to a recurrence relation if its terms satisfy the recurrence relation. For example,

$a_n = 3 * 2^n$ is a solution to the recurrence relation $a_n = a_{n-1} * 2$;

$a_n = -2^n$ is also a solution to the recurrence relation $a_n = a_{n-1} * 2$;

$a_n = c * 2^n$ is also a solution to the recurrence relation $a_n = a_{n-1} * 2$, for any $c \in \mathbb{R}$.

An *initial condition* is a specified value for some fixed a_i (usually a_0 and/or a_1). Without initial conditions, there are usually many solutions to a recurrence relation. For example,

$a_n = 3 * 2^n$ is the only solution to the r. rel. $a_n = a_{n-1} * 2, a_0 = 3$.

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A *closed formula* for a recurrence relation is a formula generating the sequence. We call a closed formula that satisfies a recurrence relation a *solution* to that relation. (Ex: $a_n = c * 2^n$)

Going from a recurrence relation to a closed form is like calculating integrals—it is not always even possible, let alone deterministic. We learn to recognize familiar types, and look for patterns.

Geometric: If $a_n = ra_{n-1}$, then

$$a_1 = r \cdot a_0, \quad a_2 = ra_1 = r(ra_0) = r^2 a_0,$$
$$a_3 = ra_1 = r(r^2 a_0) = r^3 a_0 \ldots.$$

Claim: In general, $a_n = a_0 r^n$ for whatever constant a_0 is.

Arithmetic: $a_n = m + a_{n-1}$, then

$$a_1 = m + a_0, \quad a_2 = m + a_1 = m + (m + a_0) = 2m + a_0,$$
$$a_3 = m + a_1 = m + (2m + a_0) = 3m + a_0 \ldots.$$

Claim: In general, $a_n = nm + a_0$ for whatever constant a_0 is.

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Factorial: For $n \in \mathbb{Z}_{>0}$, we define *n factorial*, denoted $n!$, by

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1.$$

For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

For convenience, we define $0! = 1$.

Then if $a_n = na_{n-1}$, we have

$$a_1 = 1 \cdot a_0, \quad a_2 = 2a_1 = 2(1 \cdot a_0) = (2 \cdot 1)a_0,$$

$$a_3 = 3a_1 = 3((2 \cdot 1)a_0) = (3 \cdot 2 \cdot 1)a_0 \ldots.$$

Claim: In general, $a_n = n!a_0$ for whatever constant a_0 is.

You try: Exercise 12

Summations

Recall, for a sequence $\{a_n\}$, the *summation notation*

$$\sum_{i=k}^{\ell} a_i = a_k + a_{k+1} + \cdots + a_\ell$$

and

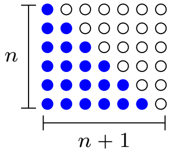
$$\sum_{i=k}^{\infty} a_i = a_k + a_{k+1} + \cdots = \lim_{\ell\to\infty} \sum_{i=k}^{\ell} a_i.$$

For example, let $a_i = i$. Define $S_n = \sum_{i=1}^{n} a_i$. Then

$$S_1 = 1, \quad S_2 = 1 + 2 = 3, \quad S_3 = 1 + 2 + 3 = 6, \ldots$$

Claim: In general, $S_n = \frac{n(n+1)}{2}$. "closed formula"

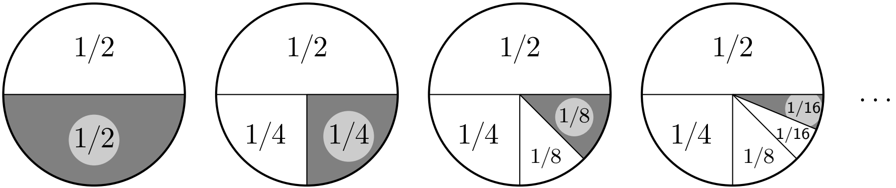
The sum S_n is half the dots in the $n \times (n+1)$ rectangle of dots:



So $\sum_{i=1}^{\infty} a_i$ is not defined (the series does not converge).

On the other hand, let $a_i = (1/2)^i$, and define $S_n = \sum_{i=1}^{n} a_i$. Then

$$S_1 = 1/2, \quad S_2 = 1/2 + 1/4 = 3/4,$$
$$S_3 = 1/2 + 1/4 + 1/8 = 7/8 = 1 - 1/8,$$
$$S_4 = 1/2 + 1/4 + 1/8 + 1/16 = 15/16 = 1 - 1/16 \ldots$$



Claim: In general, $S_n = 1 - \frac{1}{2^n}$. "closed formula"

So

$$\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \left(1 - \frac{1}{2^n}\right) = 1.$$

Solving using partial sums

The finite sum

$$S_n = \sum_{i=0}^{n} a_i = a_0 + a_1 + \cdots + a_n$$

is called the *partial sum* for the series $S = \sum_{i=0}^{\infty} a_i$. We define $S = \lim_{n\to\infty} S_n$. So to solve for S, it would be *very helpful* to get a closed form for S_n.

Example

Show

$$\sum_{i=0}^{n} cr^i = \begin{cases} c\left(\frac{r^{n+1}-1}{r-1}\right) & \text{if } r \neq 1 \\ c(n+1) & \text{if } r = 1 \end{cases}$$

using partial sums. Namely, show $rS_n = S_n + c(r^{n+1} - 1)$ and solve for S_n. Then calculate $\sum_{i=0}^{\infty} cr^i$.

Identities:

$$\sum_{i \in S} a_i + b_i = \sum_{i \in S} a_i + \sum_{i \in S} b_i \qquad \text{(addition is commutative)}$$

$$\sum_{i \in S} c * a_i = c * \sum_{i \in S} a_i \qquad \text{(distributive property)}$$

Set summations.

$$\sum_{a \in A} a \quad \text{means add up everything in } A.$$

$$\sum_{a \in A} f(a) \quad \text{means add up } f(a) \text{ for everything in } A.$$

Example:

$$\sum_{i \in \{2,4,6\}} i^2 = 2^2 + 4^2 + 6^2.$$

More notation

Double summations.

For example,

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=i}^{4} ij &= \sum_{i=1}^{3}\left(\sum_{j=i}^{4} ij\right)\\ &= \left(\sum_{j=1}^{4} 1\cdot j\right) + \left(\sum_{j=2}^{4} 2\cdot j\right) + \left(\sum_{j=3}^{4} 3\cdot j\right)\\ &= (1\cdot 1 + 1\cdot 2 + 1\cdot 3 + 1\cdot 4)\\ &\quad + (2\cdot 2 + 2\cdot 3 + 2\cdot 4) + (3\cdot 3 + 3\cdot 4)\,.\end{aligned}$$

More special summations.

Theorem

We have the following special summation identities:

$$\sum_{i=1}^{n} i = n(n+1)/2, \text{ and}$$

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r} \qquad \text{for } r \in (-1,1).$$

Notice

$$\sum_{i=1}^{\infty} i * x^{i-1} = \frac{d}{dx} \sum_{i=0}^{\infty} x^i.$$

What can we conclude?

You try: Exercise 13